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Original Article

The Role of Diffusion Rate in the Persistence of Species in Advective Homogeneous Environment

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Abstract:
We study a few single species models in a one-dimensional advective homogeneous environment. One interesting feature in these models concerns the boundary condition at the downstream end, where the species might be subject to a net loss of individuals, as measured by the parameter b which accounts for the magnitude of the loss. We investigate conditions for the persistence of a single species for general value of b, in terms of the critical habitat size and the critical diffusion rate d. Our primary objective is to investigate the relationship between the diffusion rate d and the critical habitat size L^*, specifically for $b=\infty$ and $b=1$ and, more broadly, for $b > 3/2$ and $b < 3/2$. The findings imply that for $1 \leq b < 3/2$, the essential habitat size is a decreasing function of diffusion rate, which means the diffusion rate is more larger, the species are more persister; When $3/2 < b \leq \infty$, the critical habitat size first decreases and then increases with diffusion rate increases, additionally, the species persist just on the moderate area of the diffusion rate.
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1. Introduction

Individuals in some environments are often faced with unidirectional drift (advection) that might push them out of the ecological system and thus induces the extinction of population in the habitat. Some examples are gut-dwelling bacteria [1,2,3,4], benthic marine organisms along the coastlines with long-shore currents [5], and the oases in the desert driven by wind [6, 7]. In this paper we mainly envision that organisms live in streams and rivers, where they are constantly subject to downstream drift due to water movement. We refer to [10, 11, 13, 14, 15, 16] and references therein.

How can populations persist in rivers or streams, while they are driven downstream? This question is often refereed as drift paradox in biological literatures and it has received many attentions [18, 19, 20]. Speirs and Gurney [17] proposed that diffusive movement of organisms is a key factor to promote the persistence of organisms in advective environment. More specially, they considered the following mathematical model :

$$\begin{cases} u_t = du_{xx} - \alpha u_x + u(r-u), & 0 < x < L,\ t > 0, \\ du_x(0,t) - \alpha u(0,t) = 0, & t > 0, \\ u(L,t) = 0, & t > 0, \end{cases} \#(1.1)$$

where $u(x,t)$ represents the population density at location x and time t,d is the diffusion coefficient, L is the habitat size, and $x=0$ and $x=L$ are called the upperstream end and downstream end, respectively. Parameter α is the effective water speed, and we assume that is positive as $x=L$ is the the down-stream end. The parameter $r > 0$ represents the intrinsic growth rate of the species.

We assume that d, r, α, L are all positive constants. Note that r is assumed to be a positive constant so that the environment is spatially homogeneous. Speirs and Gurney considered the stability of the trivial steady state $u=0$. They concluded that it is unstable if and only if $d > \frac{\alpha^2}{4r}$ and $L>L^*$, where

$$L^*=2d\frac{\pi-\arctan\frac{\sqrt{4dr-\alpha^2}}{\alpha}}{\sqrt{4dr-\alpha^2}}.$$

Biologically their result means that the single species can persist if and only if the advection rate is small and the habitat is sufficiently long. Does such prediction still hold for more general situations? A little surprisingly the answer is affirmative. Consider the problem.

$$\begin{cases} u_t=du_{xx}-\alpha u_x+u(r-u), & 0<x<L,\ t>0,\\ du_x(0,t)-\alpha u(0,t)=0, & t>0,\\ u(L,t)-\alpha u(L,t)=-b\alpha u(L,t), & t>0.\end{cases} \quad \#(1.2)$$

At the upstream end, the density satisfies the no-flux boundary condition, which means that effectively there is no individuals passing through this part of the boundary. At the downstream end, the parameter b measures the loss rate of individuals with respect to the flow rate [12]. The case $b=0$models the sinking, self-shading phytoplankton dynamics, see [8, 9, 21]. For $b=1$, we obtain the free-flow condition, also referred as the Danckwerts condition [22], When $L\to\infty$, formally we obtain the hostile conditions [17]. Hence, the case of zero Dirichlet boundary condition $u(L,t)=0$ can be regarded as $b=+\infty$.

When $b=1$, Vasilyeva and Lutscher [22] showed that the single species can persist if and only if $d>\frac{\alpha^2}{4r}$ and $L>L^*$, where.

$$L^*=\begin{cases} 2d\frac{\arctan\frac{\sqrt{4dr-\alpha^2}}{2rd-\alpha^2}}{\sqrt{4dr-\alpha^2}}, & \text{for } d\ge\frac{\alpha^2}{2r},\\ 2d\frac{\pi+\arctan\frac{\sqrt{4dr-\alpha^2}}{2rd-\alpha^2}}{\sqrt{4dr-\alpha^2}}, & \text{for } \frac{\alpha^2}{4r}<d<\frac{\alpha^2}{2r}.\end{cases} \quad \#(1.3)$$

However, when $b=0$, it is easy to show that for any $\alpha, L>0$, problem (1.2) admits a unique positive steady state which is globally asymptotically stable among all non- negative and not identically zero initial data. That is, a bit surprisingly, the critical rates disappeared.

To synthesize these results, for the case $b>0$, Lou and Zhou [23] proved that the species can persist if and only $d\ge\frac{\alpha^2}{4r}$ and $L>L^*$, where for $0<b<1/2$,

$$L^*=\begin{cases} 2d\frac{\arctan\frac{b\alpha\sqrt{\alpha^2-4dr}}{2rd-b\alpha^2}}{\sqrt{\alpha^2-4dr}}, & \text{as } d\ge\frac{\alpha^2}{4r},\\ \ln\frac{2dr-b\alpha^2+b\alpha\sqrt{\alpha^2-4dr}}{2dr-b\alpha^2-b\alpha\sqrt{\alpha^2-4dr}}\cdot\frac{d}{\alpha^2-4dr}, & \text{as } \frac{\alpha^2-4dr}{r}<d<\frac{\alpha^2}{4r},\end{cases} \quad \#(1.4)$$

for $b=1/2$

$$L^*=2d\frac{\arctan\frac{b\alpha\sqrt{\alpha^2-4dr}}{2rd-\alpha^2}}{\sqrt{\alpha^2-4dr}},\#(1.5)$$

and for $b>1/2$

$$L^*=\begin{cases}2d\frac{\arctan\frac{b\alpha\sqrt{4dr-\alpha^2}}{2rd-b\alpha^2}}{\sqrt{4dr-\alpha^2}}, & \text{as } d\geq\frac{\alpha^2 b}{2r},\\ 2d\frac{\pi+\arctan\frac{b\alpha\sqrt{4dr-\alpha^2}}{2rd-b\alpha^2}}{\sqrt{4dr-\alpha^2}}, & \text{as } \frac{\alpha^2}{4r}<d<\frac{\alpha^2 b}{2r}.\end{cases}\#(1.6)$$

In other words, Lou and Zhou proved that the transition occurs at b=1/2. Namely, if $b>1/2$, the persistence happens when $d>\alpha^2/4r$ and $L>L^*$ for some positive number L^*. If $0<b<1/2$, the persistence occurs when $d>\alpha^2 b(1-b)/r$and $L>L^*$As , $b\to 0$, $\alpha^2 b(1-b)/r\to 0$ and $L^*\to 0$, which is consistent with the case b=0. In particular, they proved that L^* is a strictly decreasing function of d provided that $b\in(0,1)$. This suggests that when $b\in(0,1)$, the persistence is more likely if we increase the diffusion rate. One main question is: How does L^* depend on d for general values of b?

This paper is organized as follows. In section 2 we focus on the global dynamics of model (1.2) for the special b=1, determine the persistence of a single species. In section 3 we provide the relationship between the critical habitat size and diffusion rate of the model (1.2). In section 4, we give the simulations of model (1.2) with the condition of $b>3/2$ and $b<3/2$ by taking random α, b, r. Then we discuss the main results.

2. Persistence of single specie for b=1

In this section we investigate the dynamics of the single species model (1.2) when b=1. To this end, we study the steady states of

$$\begin{cases}u_t=du_{xx}-\alpha u_x+u(r-u), & 0<x<L,\ t>0,\\ du_x(0,t)-\alpha u(0,t)=0, & t>0,\\ u_x(L,t)=0, & t>0.\end{cases}\#(2.1)$$

The critical habitat size L^* formula for b=1 has been derived by Vasilyeva and Lutscher [22],

$$L^*(d)\triangleq\begin{cases}2d\frac{\arctan\frac{\alpha\sqrt{4rd-\alpha^2}}{2rd-\alpha^2}}{\sqrt{4rd-\alpha^2}}, & \text{as } d\geq\frac{\alpha^2}{2r},\\ 2d\frac{\pi+\arctan\frac{\alpha\sqrt{4rd-\alpha^2}}{2rd-\alpha^2}}{\sqrt{4rd-\alpha^2}}, & \text{as } \frac{\alpha^2}{4r}<d<\frac{\alpha^2}{2r}.\end{cases}\#(2.2)$$

More generally, from Lou and Zhou [23], we can know for $b>½$

$$L^*=\begin{cases}2d\frac{\arctan\frac{b\alpha\sqrt{4rd-\alpha^2}}{2rd-b\alpha^2}}{\sqrt{4rd-\alpha^2}}, & \text{as } d<\frac{\alpha^2 b}{2r},\\ 2d\frac{\pi+\arctan\frac{b\alpha\sqrt{4rd-\alpha^2}}{2rd-b\alpha^2}}{\sqrt{4rd-\alpha^2}}, & \text{as } d\geq\frac{\alpha^2 b}{2r}.\end{cases}\#(2.3)$$

For national simplicity, let us denote

$$f(d)=2d\frac{\arctan\frac{b\alpha\sqrt{4rd-\alpha^2}}{2rd-b\alpha^2}}{\sqrt{4rd-\alpha^2}}.$$

Then (2.3) can be noted that

$$L^*(d)\triangleq\begin{cases} f(d), & \text{as } d\geq\frac{\alpha^2 b}{2r}, \\ \frac{2d\pi}{\sqrt{4rd-\alpha^2}}+f(d), & \text{as } \frac{\alpha^2}{4r}<d<\frac{\alpha^2 b}{2r}. \end{cases}$$

Lemma 2.1. $L^*(d)$ of (2.3) is a continuous function of d.
Proof. By the expression of (2.3), we easily know that $L^*(d)$ is continuous in the area $\frac{\alpha^2}{4r}< d <\frac{\alpha^2 b}{2r}$ and $d \geq \frac{\alpha^2 b}{2r}$. To obtain the conclusion, we just verify that the left limit equals right limit of $L^*(d)$ at the point $d=\frac{\alpha^2 b}{2r}$.

One hand, when $d \geq \frac{q^2 b}{2r}$, we have $L^*(d)=f(d)$, so

$$\begin{aligned}\lim_{d\to\left(\frac{\alpha^2 b}{2r}\right)^+} f(d) &= \lim_{d\to\left(\frac{\alpha^2 b}{2r}\right)^+} 2d\frac{\arctan\frac{b\alpha\sqrt{4rd-\alpha^2}}{2rd-b\alpha^2}}{\sqrt{4rd-\alpha^2}} \\ &= \lim_{d\to\left(\frac{\alpha^2 b}{2r}\right)^+} 2d\frac{\frac{\pi}{2}}{\sqrt{4rd-\alpha^2}} \\ &= \frac{\alpha^2 b\pi}{2r\sqrt{2\alpha^2 b-q^2}}\end{aligned}$$

On the other hand, when $\frac{\alpha^2}{4r} < d < \frac{\alpha^2 b}{2r}$, we have $L^*(d)=2d\frac{\pi}{\sqrt{4rd-\alpha^2}}+f(d)$, then

$$\begin{aligned}\lim_{d\to\left(\frac{\alpha^2 b}{2r}\right)^-} 2d\frac{\pi}{\sqrt{4rd-\alpha^2}}+f(d) &= \frac{\alpha^2 b\pi}{r\sqrt{2\alpha^2 b-\alpha^2}}+\lim_{d\to\left(\frac{\alpha^2 b}{2r}\right)^-} 2d\frac{\arctan\frac{b\alpha\sqrt{4rd-\alpha^2}}{2rd-b\alpha^2}}{\sqrt{4rd-\alpha^2}} \\ &= \frac{\alpha^2 b\pi}{r\sqrt{2\alpha^2 b-q^2}}-\frac{\alpha^2 b\pi}{2r\sqrt{2\alpha^2 b-q^2}} \\ &= \frac{\alpha^2 b\pi}{2r\sqrt{2\alpha^2 b-\alpha^2}}\end{aligned}$$

Thus, we have

$$\lim_{d\to\left(\frac{\alpha^2 b}{2r}\right)^+} L^*(d)=\lim_{d\to\left(\frac{\alpha^2 b}{2r}\right)^-} L^*(d).$$

The Conclusion is Obtained.

The first result of this subsection is as follows:

Proposition 2.2. $L^*(d)$ in (2.2) is a decreasing function of d provided b=1.
Proof. It is easy to check that $L^*(d)$ is continuous at $d=\alpha^2/2r$. Hence, to establish the proposition, we only have to show $L^*(d)$ is decreasing in both $\left(\frac{\alpha^2}{4r},\frac{\alpha^2}{2r}\right)$ and $\left(\frac{\alpha^2}{2r},+\infty\right)$. For $d\geq\alpha^2/2r$, a direct calculation gives

$$(4rd-\alpha^2)L^{*'}(d)=(4rd-\alpha^2)f'(d)$$
$$=f_1=2\frac{2rd-\alpha^2}{\sqrt{4rd-\alpha^2}}\cdot\arctan\frac{b\alpha\sqrt{4dr-\alpha^2}}{2rd-b\alpha^2}+\alpha\frac{b(1-b)\alpha^2-2brd}{dr+b(1-b)\alpha^2}$$

Regard f_1 as a function of b. By some computations one attains

$$f_1'(b)=2\frac{2rd-\alpha^2}{\sqrt{4dr-\alpha^2}}\cdot\frac{\frac{\alpha\sqrt{4rd-\alpha^2}(2rd-b\alpha^2)+b\alpha^3\sqrt{4rd-\alpha^2}}{2rd-b\alpha^2}}{1+\frac{b^2\alpha^2(4rd-\alpha^2)}{(2rd-b\alpha^2)^2}}$$
$$+\alpha\frac{[(1-2b)\alpha^2-2rd][rd+b(1-b)\alpha^2]+b(1-2b)\alpha^2[(1-b)\alpha^2-2rd]}{(dr+b(1-b)\alpha^2)^2}$$
$$=\frac{\alpha(2rd-\alpha^2)}{dr+b(1-b)\alpha^2}+\alpha rd\frac{\alpha^2\left(1+2b^2-2b\right)-2rd}{(rd+b(1-b)\alpha^2)^2}$$
$$=\frac{\left(b^2-b\right)\alpha^3(4rd-\alpha^2)}{(rd+b(1-b)\alpha^2)^2}$$

From above the calculation, we can infer

as $b<1$, then $f_1'(b)<0$;
as $b>1$, then $f_1'(b)>0$;
as $b=1$, $b=0$, then $f_1'(b)=0$.

Thus we can know b=1 is unique minimum point of $f_1'(b)$, then $f_1(b)<f_1(0)$, which immediately implies

$$L^{*'}(d)<0 \text{ for } d\geq\frac{\alpha^2}{2r}. \#(2.4)$$

For $\frac{\alpha^2}{4r}<d<\frac{\alpha^2 b}{2r}$, a direct calculations have

$$(4dr-\alpha^2)L^{*'}(d)=2\frac{2rd-\alpha^2}{\sqrt{4rd-\alpha^2}}\pi+(4rd-\alpha^2)f'(d)$$
$$=2\frac{2rd-\alpha^2}{\sqrt{4rd-\alpha^2}}\pi+f_1=f_2+f_1$$

As b=1, we know $f_2\leq 0, f_1<0$, thus we have

$$L^{*'}(d)<0 \text{ for } \frac{\alpha^2}{4r}<d<\frac{\alpha^2}{2r}. \#(2.5)$$

Combine (2.4) with (2.5), the proof is complete.

Remark 2.3. The monotonic of $L^*(d)$ with respect to d indicates that large diffusion rate increase the likelihood of the persistence of a single species as b=1. Moreover, the species survive when d is big than a certain threshold.

Remark 2.4. It is natural to ask whether the critical habitat size L^* is still a monotonically decrease function of d when b>1. It turns out that this situation is complicated,we will explore the situation in the following two sections.

3. Infinity loss rate of individuals

The infinity model is identical model (1.1), i.e.

$$\begin{cases} u_t=du_{xx}-\alpha u_x+u(r-u), & 0<x<L,\ t>0 \\ du_x(0,t)-\alpha u(0,t)=0, & t>0 \\ u(L,t)=0, & t>0 \end{cases} \#(3.1)$$

And we know, for this case, L^* was derived in Speirs and Gurney[15]:

$$L^*=2d\frac{\pi-\arctan\frac{\sqrt{4rd-\alpha^2}}{\alpha}}{\sqrt{4rd-\alpha^2}}\#(3.2)$$

Note that the domain of L^*, as a function of d, is $d>\frac{\alpha^2}{4r}$.

We main discuss the transformation law of L^* depending on d in this section.

Theorem 3.1. There exists some d_0 such that if $d>d_0$,$L^*(d)$ defined above is an decreasing function of d; if $d\le d_0$, $L^*(d)$ defined above is a increasing function of d.

Proof. For $d>\frac{\alpha^2}{4r}$, derivative to d, we have

$$g_1=(4rd-\alpha^2)L^{*'}(d)=2\frac{2rd-\alpha^2}{\sqrt{4rd-\alpha^2}}\left[\pi-\arctan\frac{\sqrt{4rd-\alpha^2}}{\alpha}\right]-\alpha\#(3.1)$$

Then g_1 derivative to d , we obtain

$$g_2=(4rd-\alpha^2)g_1'(d)=2\frac{8rd}{\sqrt{4rd-\alpha^2}}\left[\pi-\arctan\frac{\sqrt{4rd-\alpha^2}}{\alpha}\right]-\frac{\alpha(2rd-\alpha^2)}{d}.\#(3.2)$$

With the continuous of g_1 regards to d, when $d\to\infty$, $g_1=\infty$, when $d\to\frac{\alpha^2}{2r}$,$g_1=-\alpha$, then the equation $g_1=0$ must have solutions. In other words, there exists $d_0\in\left(\frac{\alpha^2}{2r},\infty\right)$ such that $g_1(d)=0$, i.e.

$$\frac{\pi-\arctan\frac{\sqrt{4rd_0-\alpha^2}}{\alpha}}{\sqrt{4rd_0-\alpha^2}}=\frac{\alpha}{4rd_0-2\alpha^2},\#(3.3)$$

due to $\frac{\pi-\arctan\frac{\sqrt{4rd_0-\alpha^2}}{\alpha}}{\sqrt{4rd_0-\alpha^2}}>0$, we have $d_0>\frac{\alpha^2}{2r}$. Then put (3.3) into (3.2) , we can get

$$
\begin{aligned}
g_2 &= \frac{8rd_0\alpha}{4rd_0-2\alpha^2} - \frac{\alpha(2rd_0-\alpha^2)}{d_0} \\
&= \alpha \frac{8r^2d_0^2-(2rd_0-\alpha^2)(4rd_0-2\alpha^2)}{(4rd_0-2\alpha^2)d_0} \\
&= 2\alpha^3 \frac{(4rd_0-\alpha^2)}{(4rd_0-2\alpha^2)d_0} \\
&> 0
\end{aligned}
$$

Thus there exists d_0 such that $g_1(d_0)=0$ and $g_2(d_0)>0$, which indicate d_0 is the minimal point of $L^*(d)$. Next we prove the minimal point is unique, there are two situations:
(1) If there are two adjacent zero points $d_0, d_1 \in \left(\frac{\alpha^2}{2r}, \infty\right)$ of g_1, i.e. $g_1(d_0)=0, g_1(d_1)=0$. Due to the expression of g_2, we easily know $g_2(d_0)>0$, $g_2(d_1)>0$. Thus we obtain that adjacent points d_0, d_1 are the minimal points, which is contradicted.
(2) If there exists another zero point $d_1 \in \left(\frac{\alpha^2}{4r}, \frac{\alpha^2}{2r}\right)$, $2rd_1-\alpha^2<0$, with the expression of (3.1), we have g_1 always less than 0 . This is contradicted.

So g_1 has unique zero point We can then conclude: $\frac{\alpha^2}{4r}<d<d_0$, $L^*(d)$ is a decreasing function of d; $d>d_0$, $L^*(d)$ is a increasing function of d. The proof is complete.

Remark 3.2. The proof of Theorem 3.2 confirms Lou[12]'s conjecture that when the net loss of species is large, the critical habitat length and dispersal rate no longer maintain their monotonicity. When the diffusivity is less than d_0, the critical habitat length L^* is still a decreasing function of d, and as d increases, the critical habitat length L^* decreases. If $L> L^*$, the species will survive, and the increase in the diffusion rate increases the likelihood of survival. When the diffusivity d is greater than d_0, the diffusivity is no longer "better" as larger, and the critical habitat length is an increasing function of the diffusivity. The larger the d, the larger the corresponding L^*, the less likely the $L>L^*$ is, and the more difficult it is for the species to survive, indicating that the greater the diffusion rate at this time, the more likely it is to survive the species.

When $b=+\infty$, the model studied in Speirs [17], we know $u=0$ is the local stability of steady state for $d \le \frac{\alpha^2}{4r}$, The steady state has a stable positive solution if and only if $d > \frac{\alpha^2}{4r}$ and $L > L^*$. If we fix other parameters but let d and L vary, then we have the following results:

Theorem 3.3. There exists some positive value L^*, which is independent of d and L, such that:
(1) If $L < L^*$, then the single species can not persist for any d.
(2) If $L > L^*$, then there exist two positive constants d_*, d^* such that the species can persist if and only if d falls in the interval (d_*, d^*). In other words, the species can persist if and only if diffusion lies in an intermediate range.

Proof. To the first situation, we have proved that d_0 is the unique minimal point of $L^*(d)$, the corresponding critical habitat length L^*, if $L<L^*$, we have $\lambda_1>0$, according to solution stability theorem, we know the single species can't persist.

By the contradiction to the second situation, i.e., $\forall d > \frac{\alpha^2}{4r}$, we have $L > L^*$. First of all,

$$
L^*=2d\frac{\pi-\arctan\frac{\sqrt{4rd-\alpha^2}}{\alpha}}{\sqrt{4rd-\alpha^2}}
$$

with the above expression, we know

$$
\lim_{d\to\frac{\alpha^2}{4r}} L^*=+\infty,\ \lim_{d\to\infty} L^*=+\infty. \#(3.4)
$$

When d sufficiently close to $\frac{\alpha^2}{4r}$, the critical habitat size is very large, by the assumption L is limited, otherwise it has no significance. By Theorem 3.1, $L<L^*$ for d sufficiently close to $\frac{\alpha^2}{4r}$. Similarly, if d sufficiently large, we can deduce $L<L^*$ for sufficiently large d, there is a contradiction. So the species can persist if and only if in an intermediate range.

Remark 3.4. Theorem 3.3 means that once this intermediate zone is exceeded, the species will become extinct.

Remark 3.5. For $b=\infty$, L^* is no longer monotonic with respect to d, which indicates that the diffusion rate over an area will reduce the likelihood of species survival. Diffusion rate beyond this range will help increase the likelihood of survival of the species. This property is completely different from $b=1$.

4. The relation of L^* and d when $1<b<\infty$

First of all, the mathematical model is given by

$$\begin{cases} u_t=du_{xx}-\alpha u_x+u(r-u)=0, & 0<x<L,\ t>0, \\ du_x(0,t)-\alpha u(0,t)=0, & t>0, \\ u(L,t)-\alpha u(L,t)=-b\alpha u(L,t), & t>0. \end{cases} \quad \#(4.1)$$

According to Lou and Zhou [23], the critical rate L^* is given and for $b>1/2$

$$L^*=\begin{cases} 2d\dfrac{\arctan\dfrac{b\alpha\sqrt{4dr-\alpha^2}}{2rd-b\alpha^2}}{\sqrt{4dr-\alpha^2}}, & \text{as } d\geq\dfrac{\alpha^2 b}{2r}, \\ 2d\dfrac{\pi+\arctan\dfrac{b\alpha\sqrt{4dr-\alpha^2}}{2rd-b\alpha^2}}{\sqrt{4dr-\alpha^2}}, & \text{as } \dfrac{\alpha^2}{4r}<d<\dfrac{\alpha^2 b}{2r}. \end{cases} \quad \#(4.1)$$

In this section we main discuss the property of $L^*(d)$ when $1<b<+\infty$.

Lemma 4.1. (Lou and Zhou [23]) Given any $\alpha>0$, $r>0$, and $d>0$, $b=3/2$ is a critical value which can transform the trend L^* depending on d.

Proof. By direct calculation, we have

$$\begin{aligned}(4rd-\alpha^2)L^{*'}(d)&=2\frac{2rd-b\alpha^2}{\sqrt{4rd-\alpha^2}}\arctan\frac{b\alpha\sqrt{4rd-\alpha^2}}{2rd-b\alpha^2}+\alpha\frac{b(1-b)\alpha^2-2brd}{rd+b(1-b)\alpha^2} \\ &=2\frac{2rd-b\alpha^2}{\sqrt{4rd-\alpha^2}}\left[\frac{b\alpha\sqrt{4rd-\alpha^2}}{2rd-b\alpha^2}-\frac{b^3\alpha^3\left(\sqrt{4rd-\alpha^2}\right)^3}{(2rd-b\alpha^2)^3}\right] \\ &\quad+\alpha\frac{b(1-b)\alpha^2-2brd}{rd+b(1-b)\alpha^2}+o\left(\frac{1}{d^2}\right) \\ &=2\frac{b^3\alpha^3}{r}\left[\frac{2}{3}b-1\right]\cdot\frac{1}{d}o\left(\frac{1}{d^2}\right)\end{aligned}$$

Clearly, if $b>3/2$, then for sufficiently large d, $L^{*'}(d)>0$; while if $b<3/2$, then $L^{*'}(d)<0$ for sufficiently large d.

According to the Lemma 4.1 the range of b should be divided into two parts to inspect the monotonicity of L^* : One is $1<b<3/2$, and the other is $3/2<b<+\infty$. This section mainly relies on MATLAB simulations to illustrate the monotonicity of L^*, by fixing α,r,b by random function, and depict the picture of d and L^*. Through numerous simulations, we can intuitively conclude the relationships from the graphic.

Conjecture 1. As $1<b\leq\frac{3}{2}$, L^* is a decreasing function of d.

In the area $1 < b \leq \frac{3}{2}$, by fixing α, r, b by random function, the figures are followed:

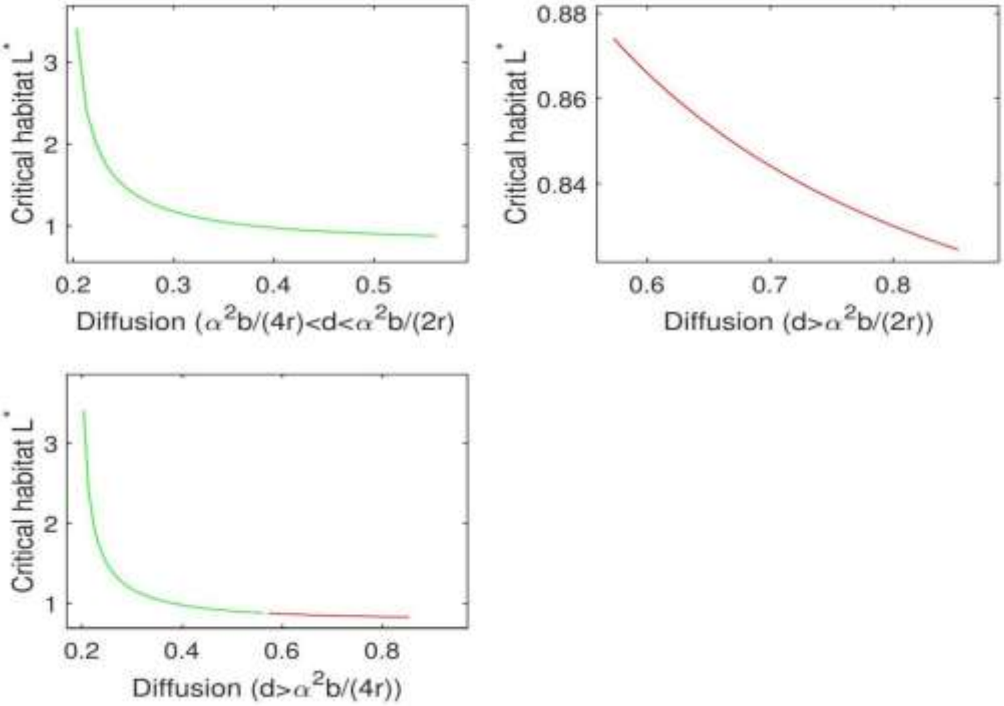

Figure 1. In the range of $1 < b \leq \frac{3}{2}$, the x -axis is d, the y -axis is L^*. The first figure is $d\in\left(\frac{\alpha^2}{4r},\frac{\alpha^2 b}{2r}\right)$, L^* is decreasing; The second figure is $d > \frac{\alpha^2 b}{2r}$, L^* is also decreasing; The third figure combined the first two.

As can be seen from Figure 1, when $1 < b \leq \frac{3}{2}$, $\forall d > \frac{\alpha^2}{4r}$, L^* is decreasing with d.

Theorem 4.2. Given any $r > 0$, $\alpha > 0$, $1 < b \leq \frac{3}{2}$, then

$$\lim_{d\to\frac{\alpha^2}{4r}} L^*=+\infty,\ \lim_{d\to\infty} L^*=\frac{bq}{r} \#(4.2)$$

and L^* is arranged in $\left(\frac{bq}{r}, \infty\right)$.

Proof. When d tends to both ends of range, find the corresponding limit of L^*. By Lou [23], when $\frac{\alpha^2}{4r} < d < \frac{\alpha^2}{2r}$, we have

$$L^*(d)=2d\frac{\pi+\arctan\frac{b\alpha\sqrt{4rd-\alpha^2}}{2rd-b\alpha^2}}{\sqrt{4rd-\alpha^2}},$$

let d tends to $\frac{\alpha^2}{4r}$,

$$\begin{aligned}\lim_{d\to\frac{\alpha^2}{4r}} L^* &= \lim_{d\to\frac{\alpha^2}{4r}} 2d\frac{\pi+\arctan\frac{b\alpha\sqrt{4dr-\alpha^2}}{2rd-b\alpha^2}}{\sqrt{4dr-\alpha^2}}\\ &= \lim_{d\to\frac{\alpha^2}{4r}} 2d\frac{\pi+\frac{b\alpha\sqrt{4dr-\alpha^2}}{2rd-b\alpha^2}}{\sqrt{4dr-\alpha^2}}\\ &= \lim_{d\to\frac{\alpha^2}{4r}} 2d\left[\frac{\pi}{\sqrt{4dr-\alpha^2}}+\frac{2rd\alpha}{2rd-b\alpha^2}\right]\\ &=+\infty\end{aligned}$$

When $d\to\infty$,

$$\lim_{d\to\infty} L^* = \lim_{d\to\infty} 2d\frac{\arctan\frac{b\alpha\sqrt{4dr-\alpha^2}}{2rd-b\alpha^2}}{\sqrt{4dr-\alpha^2}} = \lim_{d\to\infty} 2d\frac{\frac{b\alpha\sqrt{4dr-\alpha^2}}{2rd-b\alpha^2}}{\sqrt{4dr-\alpha^2}} = \frac{b\alpha}{r}$$

Through numerical simulation, we have L^* decreases monotonically with respect to d when $1 < b \le \frac{3}{2}$, so the range of L^* is $\left(\frac{\alpha b}{r},\infty\right)$.

Remark 4.3. The monotonicity of L^* with respect to d for $b\in\left[1,\frac{3}{2}\right)$ indicates that large diffusion rate increases the likelihood of the persistence of a single species, i.e., there exists a non-negative steady state with $L>L^*$. If the habitat size less than $\frac{b\alpha}{r}$, regardless of the value of d, the species will not survive. If the diffusion rate $d\to\frac{\alpha^2}{4r}$, then $L^*\to\infty$, which indicates the diffusion rate is small, the species will distinct.

Conjecture 2. As $b>3/2$, at first L^* decreases, and then increases.

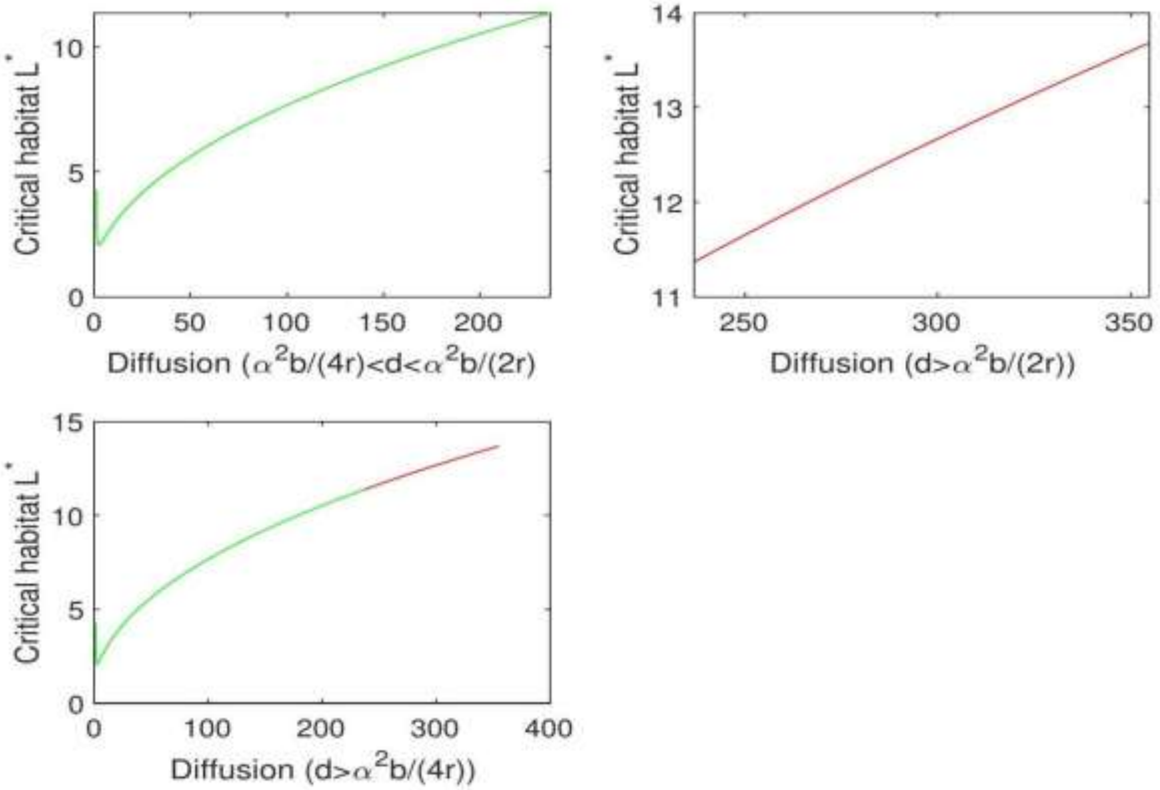

Figure 2. In the range of $b > \frac{3}{2}$, the x -axis is diffusion rate d, the y -axis is critical habitat size L^*. The first figure is $d\in\left(\frac{\alpha^2}{4r},\frac{\alpha^2 b}{2r}\right)$, L^* is decreasing; The second figure is $d > \frac{\alpha^2 b}{2r}$, L^* is also decreasing; The third figure combined the first two.

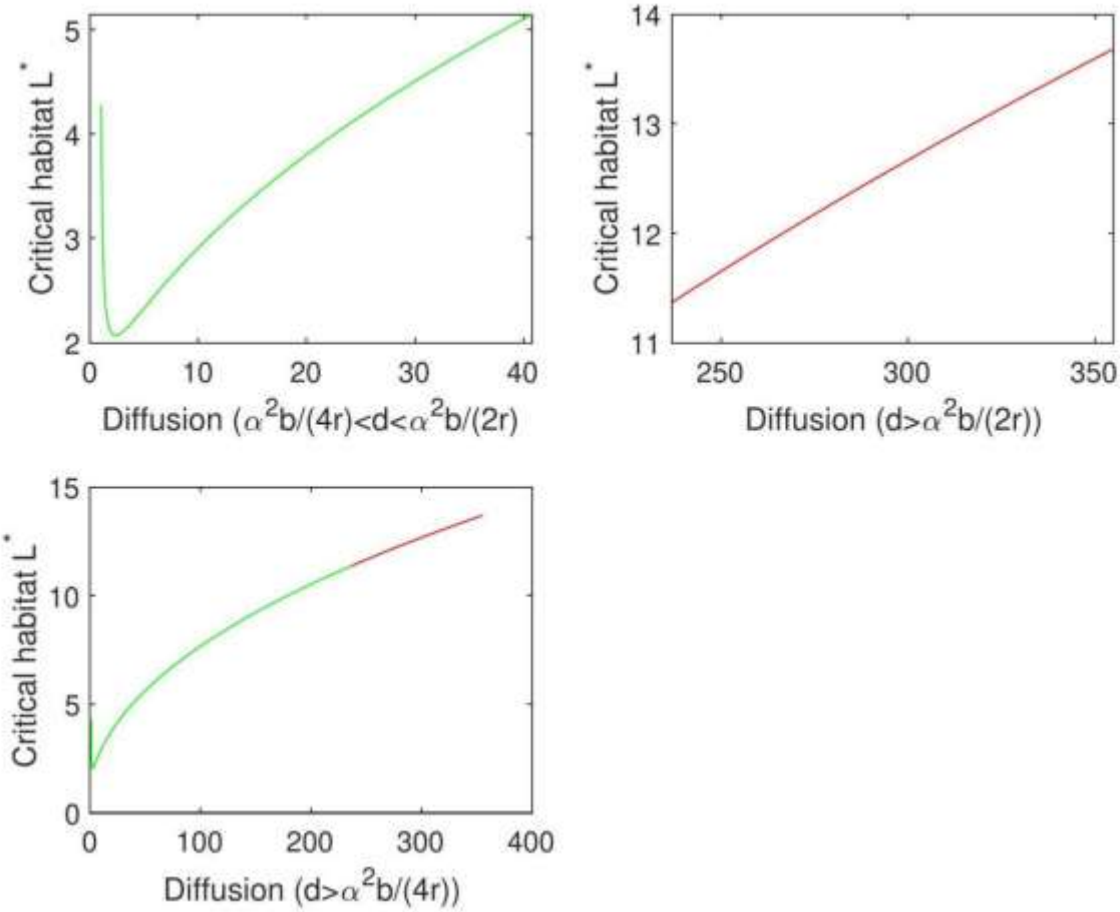

Figure 3. In the range of $b > \frac{3}{2}$, the x -axis is diffusion rate d, the y -axis is critical habitat size L^*.

The Figure 3 is a zoomed-in view of Figure 2, and it can be clearly seen that in Figure 3, the critical habitat size L^* decreases first and then increases. This suggests that when species loss is large, it is not that the larger the dispersal rate, the easier it is for species to survive. It can be seen that species survive if and only if the dispersal is in an intermediate region. Once this area is exceeded, the dispersal rate is counterproductive to the survival of the species, and the greater the dispersal rate, the more difficult it is for the species to survive and the longer the length of residence required.

Remark 4.4. When b takes different situations, integrate the relationship between L^* and d..
(1) When $0< b < \frac{3}{2}$, L^* is a monotonically decreasing function of d, i.e., the diffusion rate is conducive to the survival of the species, and the greater the diffusion rate, the greater the probability of the species surviving. If $d > \frac{\alpha^2}{4r}$, when $L > L^*$, the model (1.2) has a unique global positive equilibrium point, otherwise, $u\equiv 0$ is a stable equilibrium point.
(2) When $\frac{3}{2} < b \leq \infty$, L^* and d no longer maintain monotonicity. That is, there are normal numbers d^* and d_*, and the species can only survive if the diffusivity falls within this range. In other words, both too low and too high a dispersion rate reduce the likelihood of a species surviving.

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