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Original Article

Teaching Research on Advanced Macroeconomic Modeling from the Perspective of "Solving Equations"

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Abstract:
In the study of economics, advanced macroeconomic modeling has a high learning threshold due to its difficulty and the breadth of knowledge required. This paper uses the thread of "solving equations" to "connect" the main theoretical models of advanced macroeconomics from simple to complex, from the Solow model to the New Keynesian model. It is clear that when new endogenous variables are introduced into the basic model, new equations can be introduced either through the first-order conditions of optimal planning or through market clearing methods, ensuring that the number of endogenous variables is equal to the number of equations. Mastering the above expansion methods not only effectively reduces the threshold for learning advanced macroeconomic theory modeling but also helps students to form new model designs based on research backgrounds in future economic research, thereby improving the theoretical depth of research.
Keywords: Macroeconomics; Solving Equations; Teaching Research; Theoretical Modeling

1. Introduction

In the current study of advanced macroeconomics in China, what students find difficult is how to build macroeconomic models. Even with a grasp of basic static and dynamic optimization techniques, the derivation process involves a large number of intermediate equations, and which equations will enter the final system of difference equations still puzzles students. At the same time, we have noticed that the econometric techniques frequently used in economic research is constantly being updated and replaced. In the past, models such as cointegration and threshold regression in time series analysis were more common, and recently, micro-econometric techniques such as difference-in-differences, regression discontinuity, and matching in causal inference have been rapidly applied (Cunningham, 2021). However, the use of theoretical modeling tools in papers has always been lukewarm. An important reason is that theoretical modeling has a high learning threshold, and mastering this technology requires continuous time investment. Undergraduate papers using micro-econometric methods to argue have been published in academic journals from time to time, but undergraduate papers using theoretical modeling to argue are rare in China. In fact, attentions to theoretical modeling has been increasing (Li Rong et al., 2022; Wang Ziqi et al., 2024). So, how to teach a thread of clues, along which, from simple to complex, can help students master this important theoretical modeling tool in less time, is of great significance for the high-quality training of macroeconomic talents.

The content of theoretical modeling mainly includes four parts, and this paper only discusses one part. Specifically, theoretical modeling is the mathematicalization of economic ideas. The process of mathematicalization usually involves three types of variables: endogenous variables, exogenous variables, and parameters. Endogenous variables are variables determined by the equation system, for example, if what is to be solved is, then is an endogenous variable; if what is to be solved is, then is an endogenous variable. Exogenous variables are not determined by the equation system and are given in advance by the modeler outside the system. For difference equations, exogenous variables can be different in each period, but this difference is decided outside the system. Parameters are also decided outside the system, but they are the unchanged part in the equation system in each period, reflecting the unchanging characteristics of the economy and society, such as institutions and preferences. Through the classification and definition of economic variables, the construction of macroeconomic theoretical models can be divided into four parts: one is to obtain a nonlinear difference equation system containing three types of variables through the first-order conditions of optimal planning, resource constraints, etc. The second is to linearize the equations into linear difference equations using logarithmic linearization, based on methods such as Uhlig (1999) and Blanchard and Kahn (1980), to obtain the solution of endogenous variables. The third is to use observable macroeconomic time series data to calibrate the parameter values in the model. The fourth is to conduct comparative static analysis based on the theoretical model, or to discuss the trajectory of endogenous variables under the impact of exogenous variables, thus providing valuable reference suggestions for the implementation of economic policies. Due to space limitations, this paper only focuses on the first step, that is, how to obtain the appropriate difference equation system according to one's own economic ideas through consumer utility maximization, producer profit maximization, etc.

Advanced macroeconomics textbooks have different focuses on the above four parts. For example, Heer and Maussner (2024) focus more on the second part, that is, model solving. They discuss in-depth the local and global solution methods of endogenous variables in the model and provide corresponding execution codes. DeJong and Dave (2011) focus more on the third part, that is, model calibration. They discuss in detail the generalized method of moments estimation, simulation method of moments estimation, maximum likelihood estimation, and Bayesian estimation of model parameters based on three basic theoretical models. Benassy-Quere et al. (2010) focus on the fourth part, detailing the theory of fiscal policy, monetary policy, exchange rate policy, growth policy, tax policy, and their application in the real world. Similar to this paper's focus on the first part, which is committed to building theoretical equations from economic ideas, is the classic textbook of advanced macroeconomics by Romer (2019). This textbook uses a large number of mathematical formulas based on static and dynamic optimization theory to complete the main exposition of classic macroeconomic theories. In terms of teaching arrangements, they start with the Solow model, and then elaborate on overlapping generations, endogenous growth, real business cycles, up to the New Keynesian model. Both in the construction of model details and the exposition of economic implications, they have a deep insight. This teaching arrangement is similar in many advanced macroeconomic textbooks (Blanchard and Fischer, 1989; Heijdra, 2017; McCandless, 2008). The key point of their writing is to clearly, accurately, and rigorously elaborate economic ideas. Theoretical models only exist to serve economic ideas. For students, the advantage is that it greatly facilitates their understanding of the evolution of economic ideas and the logical formation of contemporary macroeconomic theories. The disadvantage is also obvious. Because the mathematical expression of theoretical models lacks an internal connecting thread, students are often confused by the various changes in the mathematical expression of different theoretical models. They lack sufficient control over how the construction of theoretical models can be realized from 0 to 1, and how different economic theories can be transformed between each other.

The innovation of this paper lies in sorting out a teaching thread that connects different economic theoretical models, making up for the shortcomings of theoretical model construction teaching in current advanced macroeconomics textbooks. This teaching thread is "solving equations." For a system of linear equations, the existence and uniqueness of solutions not only require the number of equations to be equal to the number of variables but also require the determinant formed by the variable coefficients not to be 0. In other words, the number of independent equations must be equal to the number of variables. Advanced macroeconomic models are often composed of nonlinear difference equations, and the existence of solutions is more complex (Ascher et al., 1995). However, the number of independent equations being equal to the number of variables is still an important condition for solving. Therefore, in the process of macroeconomic modeling, the guiding idea is always to find an independent equation for each endogenous variable. To introduce new endogenous variables, new independent equations must be added.

For example, the Solow model has only one endogenous variable, that is, the capital stock. Then the model also has only one equation - the resource constraint equation (see equation (1) below). When we expand it to the Real Business Cycle model (RBC), a new endogenous variable, that is, consumption, is introduced. Then a new equation - the Euler equation for consumption - needs to be added. As the model becomes more complex, the number of endogenous variables gradually increases, and the number of equations will also increase accordingly. Therefore, this paper always grasps the key point of "solving equations" and describes the construction ideas of the main models of advanced macroeconomics from the simple Solow model, RBC model, Cash in Advanced model (CIA), imperfect competition model to the complex New Keynesian model in five steps. Students can observe how to introduce new endogenous variables based on research backgrounds in the expansion of the basic Solow model, and according to research needs, they can either use first-order conditions or market clearing and other different methods to construct new equations. Instead of losing focus and being unable to exert effort or find a way in the face of numerous theoretical model equations.

The rest of this paper is arranged as follows. The second part elaborates on the single-equation Solow model. The third part internalizes the savings rate of the Solow model, introduces the RBC model, and makes two simple expansions of labor internalization and the introduction of the government sector. The fourth part introduces money and prices into the model through the constraint that consumption must use cash, that is, the Cash in Advanced model. The fifth part divides the production sector into final products and intermediate products in the CIA model, introduces monopoly competition factors, that is, the imperfect competition model. The sixth part introduces price stickiness through the assumption that the prices of intermediate products cannot change rapidly, and elaborates on the construction of the New Keynesian model. The seventh part is a summary.

II. Solow Model

The Solow model originally comes from Solow (1956). It is assumed that the economy only produces one product, and the production function can be written in the Cobb-Douglas form as follows,

$$Y_t = A_t f(K_t, H_t) = A_t K_t^{\alpha} H_t^{1-\alpha}$$

where Y_t is the output of the product in period t, A_t is the level of technology in period t, K_t is the capital stock in period t, and L_t is the total labor input in period t. At the same time, the main equation of the Solow model describes the accumulation of capital, also known as the resource constraint equation, which is specifically as follows,

$$K_t = (1-\delta)K_{t-1} + I_t$$

where δ is the depreciation rate, and I_t is the investment amount. Since investment equals savings, if it is assumed that the proportion of savings in output is σ, then the above equation can be written as,

$$K_t = (1-\delta)K_{t-1} + \sigma A_{t-1} K_{t-1}^{\alpha} H_{t-1}^{1-\alpha} \quad (1)$$

The above equation is the key equation of the Solow model. In this equation, $K_t$is an endogenous variable; A_t are exogenous variables, which can either be given a value for each period or can be assumed to grow at a certain rate $A_t = (1+\gamma)A_{t-1}$, and γ are given parameters; H_t is also an exogenous variable, which can either be given a value for each period or can be assumed to grow at a certain rate $H_t = (1+n)H_{t-1}$, and n is a given growth parameter; δ, σ, α are parameters of the model.

From the perspective of solving equations, the number of equations equal to the number of endogenous variables is an important condition for solving. The Solow model is the simplest model in macroeconomics because it has only one equation and one endogenous variable. In fact, solving such a differential equation is not complicated. Given the initial value of the endogenous variable, the solution of the endogenous variable for any period can be obtained using equation (1).

For consistency with subsequent discussions, the equation can also be written in per capita form. Dividing both sides of the above equation (1) by H_t, we get,

$$k_t = \frac{(1-\delta)k_{t-1} + \sigma A_{t-1}k_{t-1}^{\alpha}}{1+n}$$

where $k_t = \frac{K_t}{H_t}$. Below, if not specifically stated, lowercase letters usually represent per capita variables, and uppercase letters represent total variables. Many times, to facilitate the argument, it is assumed that labor growth is 0, that is $n = 0$, making the above equation simpler,

$$k_t = (1-\delta)k_{t-1} + \sigma A_{t-1}k_{t-1}^{\alpha} \tag{2}$$

III. RBC Model

3.1 Classic RBC Model

The RBC model is a simple expansion based on the Solow model (Hansen, 1985). In the Solow model, it is assumed that the savings ratio is an unchanging exogenous given constant. The RBC model attempts to internalize it. The so-called internalization means that it is no longer given exogenously but is made an endogenous variable, and a new equation is introduced to make the original Solow model with one variable and one equation become a new economic system with two variables and two equations.

The specific idea is that total output (per capita output is recorded as) is either consumed (per capita consumption is recorded as c) or saved (per capita savings is recorded as s), that is,

$$y_t = A_{t-1}k_{t-1}^{\alpha} = c_t + s_t$$

In the Solow model, k_t is an endogenous variable, so from the above equation, if c_t can be endogenously determined, it is equivalent to endogenously determining s_t, and thus endogenously determining σ_t. Therefore, we need to introduce a new decision equation for the new endogenous variable. In macroeconomics, it is usually obtained through the maximization of lifelong utility to get the decision formula for consumption. Specifically, the optimal plan is as follows,

$$Max_{c_t} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$s.t.\, k_t = (1-\delta)k_{t-1} + f(k_{t-1}) - c_t \tag{3}$$

where β is the discount factor for utility, and $u(\cdot)$ is the utility function. The above optimal plan indicates that the representative consumer, based on the capital accumulation form constraint or resource constraint, chooses the optimal consumption for each period to maximize the sum of lifelong utility. This optimal plan can be obtained using the Lagrange method to obtain the first-order condition. Specifically, construct the Lagrangian function as follows,

$$\begin{aligned} L &= u(c_0) + \beta u(c_1) + \cdots \beta^t u(c_t) + \beta^{t+1}u(c_{t+1}) + \cdots \\ &= u\big((1-\delta)k_{-1} + f(k_{-1}) - k_0\big) + \cdots \\ &+\beta^t u\big((1-\delta)k_{t-1} + f(k_{t-1}) - k_t\big) + \beta^{t+1}u\big((1-\delta)k_t + f(k_t) - k_{t+1}\big) + \cdots \end{aligned}$$

The second equality of the above equation holds by substituting the constraint equation (3). Let $\frac{\partial L}{\partial k_t} = 0$, then we have,

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta[1 - \delta + f'(k_t)] \tag{4}$$

Since all individuals are homogeneous, we can agree on the following form from individual to total,

$$K_t = \int_0^1 k_t^i di, \quad C_t = \int_0^1 c_t^i di$$

where k_t^i, c_t^i represents the capital stock and consumption of the i-th representative agent in period t, respectively (the superscript i can also be omitted because of the homogeneity of the individuals) and the individuals are a continuum on the 0 and 1 intervals. The above aggregation rule means that

$$K_t = k_t^i, \quad C_t = c_t^i$$

Therefore, equations (3) and (4) can also be directly written in total form,

$$K_t = (1 - \delta)K_{t-1} + f(K_{t-1}) - C_t$$
$$\frac{u'(C_t)}{u'(C_{t+1})} = \beta[1 - \delta + f'(K_t)]$$

The above two equations form the differential equation system on which the economic system depends. It can be seen that compared with the Solow model, the RBC model introduces a new endogenous variable, so we use the first-order condition of the optimal plan as a new equation, and finally form an economic system with two variables (K_t, C_t) and two equations. There are many ways to solve such a differential equation system, including local solution methods such as perturbation methods using Taylor expansion at the steady state, and global solution methods such as value function iteration, parameterized expectations, etc., which can refer to Heer and Maussner (2024).

3.2 Two Simple Expansions of the Classic RBC Model

3.2.1 Internalizing Labor Supply

It should be noted that the labor supply in the previous model is exogenously given. Labor supply can also be internalized, and the key is to find a dynamic equation to determine the movement form of labor supply. Usually, leisure can be introduced into the utility function, and the labor supply of workers per period is fixed and can be standardized to 1, so labor supply is $h = 1 - l$, where l is leisure. Therefore, introducing leisure is equivalent to introducing labor supply. The consumer's optimal plan can be rewritten as,

$$Max_{c_t,l_t} \sum_{t=0}^{\infty} \beta^t \bar{u}(c_t, l_t) = Max_{c_t,h_t} \sum_{t=0}^{\infty} \beta^t \bar{u}(c_t, 1 - h_t) = Max_{c_t,h_t} \sum_{t=0}^{\infty} \beta^t u(c_t, h_t)$$
$$s.t.\, k_t = (1 - \delta)k_{t-1} + f(k_{t-1}, h_t) - c_t$$

Notice that not only the objective function is changed compared to the classical RBC optimal programming, but the production function in the constraints also introduces a labor supply variable. Moreover, this optimal planning maximizes lifetime utility by choosing c_t, h_t. Therefore, two first-order conditions on c_t, h_t can be obtained using the Lagrangian method. Specifically, the Lagrangian function is constructed as follows,

$$L = u(c_0, h_0) + \beta u(c_1, h_1) + \cdots \beta^t u(c_t, h_t) + \beta^{t+1} u(c_{t+1}, h_{t+1}) + \cdots$$
$$= u\big((1-\delta)k_{-1} + f(k_{-1}, h_0) - k_0, h_0\big) + \cdots$$
$$+\beta^t u\big((1-\delta)k_{t-1} + f(k_{t-1}, h_t) - k_t, h_t\big) + \beta^{t+1} u\big((1-\delta)k_t + f(k_t, h_{t+1}) - k_{t+1}, h_{t+1}\big) + \cdots$$

Let $\frac{\partial L}{\partial k_t} = 0$ and $\frac{\partial L}{\partial h_t} = 0$. The two first order conditions are as follows.

$$\frac{u_c(c_t, h_t)}{u_c(c_{t+1}, h_{t+1})} = \beta[1 - \delta + f'(k_t, h_{t+1})] \tag{5}$$

$$\frac{u_h(c_t, h_t)}{u_c(c_t, h_t)} = -f_h(k_{t-1}, h_t) \tag{6}$$

where u_c, u_h represent the partial derivatives of the utility function with respect to consumption and labor, respectively. Similarly, because households are homogeneous, the aggregation rule for labor supply is $H_t = \int_0^1 h_t^i di$, where h_t^i is the total labor supply in period t, and $H_t = h_t^i$. Then, equations (5) and (6), along with the constraint condition, can also be directly written in total form,

$$\frac{u_c(C_t, H_t)}{u_c(C_{t+1}, H_{t+1})} = \beta[1 - \delta + f'(K_t, H_{t+1})]$$
$$\frac{u_h(C_t, H_t)}{u_c(C_t, H_t)} = -f_h(K_{t-1}, H_t)$$
$$K_t = (1-\delta)K_{t-1} + f(K_{t-1}, H_t) - C_t$$

This is an economic system with three endogenous variables C_t, H_t, K_t and three equations. Compared with the classic RBC model, we have introduced a new endogenous variable H_t and also added a new first-order condition.

3.2.2 Simple Introduction of Government Sector

We can also introduce the government sector into the above model. The government mainly affects the economy through transfer payments. If we introduce government purchases as an endogenous variable, the key is to find a decision equation for it. The basic logic of finding a new equation is to express the newly introduced variable with the original endogenous variables in the model, or to implicitly decide it through an implicit function. In the process of internalizing savings, we used the first-order condition of the optimal plan to decide consumption, thereby determining savings and expanding the Solow model; in the process of internalizing labor, we added a new first-order condition to the optimal plan to increase the decision equation for labor supply, expanding the classic RBC model. So, do we need to add a new first-order condition to internalize government purchases? In fact, there are various ways to internalize government purchases. The simplest form is to assume that government purchases (g) are always a fixed proportion of total output (σ_g), and assume that this part of government purchases does not participate in economic construction but is directly consumed, then we have,

$$g_t = \sigma_g \cdot f(k_t, h_t) \tag{7}$$

At the same time, modify the constraint condition to,

$$k_t = (1-\delta)k_{t-1} + f(k_{t-1}, h_t) - c_t - g_t \quad (8)$$

The original two first-order conditions on labor supply and consumption do not change. Then this four equations are directly converted into aggregate form, and the new economic system contains four endogenous variables C_t, H_t, K_t, G_t, and also possesses the four aggregate equations corresponding to Eqs. (5) to (8). It can be seen that the introduction of new equations is not necessarily realized by first-order conditions in optimal planning, and direct decisions similar to Eq. (7) are common.

3.2.3 Introduction of Government Departments in a Decentralized Economy

There are more realistic ways to introduce government transfer payments in the literature (McCandless, 2008), which considers that government revenue comes from taxation, rather than simply stipulating a total output ratio. If we use this method to introduce government purchases, we need to re-model the economy from the perspective of the dispersed economy (rather than the aforementioned central planner). The characteristic of the dispersed economy is that there are not only consumption departments but also production departments. There is an exchange between the consumption and production departments, so the system includes prices such as wages and interest rates. We first re-model the economy from the perspective of the *i*-th consumer, and its optimal plan can be written as,

$$Max_{c_t,h_t} \sum_{t=0}^{\infty} \beta^t u(c_t^i, h_t^i)$$

$$s.t.\, k_t^i = (1-\delta)k_{t-1}^i + w_t h_t^i + r_t k_{t-1}^i - c_t^i \quad (9)$$

where w_t is wages, and r_t are interest rates, which are both prices. The constraint condition indicates that the capital stock in the next period is the capital stock after depreciation in this period plus the labor and capital income earned in this period minus consumption. Similar to the previous calculation steps, the first-order condition about c_t^i, h_t^i can be written as,

$$\frac{u_c(c_t^i, h_t^i)}{u_c(c_{t+1}^i, h_{t+1}^i)} = \beta(1-\delta+r_t) \quad (10)$$

$$\frac{u_h(c_t^i, h_t^i)}{u_c(c_t^i, h_t^i)} = -w_t \quad (11)$$

The two first-order conditions about c_t^i, h_t^i in (10) and (11), as well as the constraint condition (9), can also be directly written in total form. It should be noted that due to the introduction of the new variable w_t, r_t, the equation system cannot be solved. Therefore, either regard w_t, r_tas an exogenous variable or internalize it. The literature usually introduces productions internalizes w_t, r_t. Specifically, from the perspective of producers, they choose the input of capital and labor factors to maximize profits, and the optimal plan is,

$$Max_{k_{t-1},h_t} f(K_{t-1}, H_t) - w_t H_t - r_t K_{t-1}$$

The first-order conditions indicate that,

$$w_t = f_h(K_{t-1}, H_t) \tag{12}$$

$$r_t = f_k(K_{t-1}, H_t) \tag{13}$$

Then, based on the individual being a continuum on the interval [0,1], we can have the aggregation condition $c_t^i = C_t, h_t^i = H_t, k_t^i = K_t$, and equations (9) to (11) can be written as,

$$K_t = (1-\delta)K_{t-1} + w_t H_t + r_t K_{t-1} - C_t \tag{14}$$

$$\frac{u_c(C_t, H_t)}{u_c(C_{t+1}, H_{t+1})} = \beta(1-\delta+r_t) \tag{15}$$

$$\frac{u_h(C_t, H_t)}{u_c(C_t, H_t)} = -w_t \tag{16}$$

So far, there are five endogenous variables, and we have five equations (12) to (16), and the model is closed.

Furthermore, if the government levies taxes on labor income and returns the tax revenue to individuals at one time, then the constraint condition (9) needs to be modified to,

$$k_t = (1-\delta)k_{t-1} + (1-\tau)\cdot w_t h_t + r_t k_{t-1} - c_t + T_t \tag{17}$$

where τ is the tax rate, and the returned tax revenue is,

$$T_t = \tau w_t H_t \tag{18}$$

It should be noted that although the two items $-\tau w_t h_t$ and T_t on the right side of equation (17) are equal in absolute value, they cannot be offset before obtaining the individual first-order conditions. Because T_t is the part of the macro level allocated to individuals, individuals cannot choose this part to optimize the objective function. Therefore, the first-order condition about h_t is,

$$\frac{u_h(c_t^i, h_t^i)}{u_c(c_t^i, h_t^i)} = -(1-\tau)w_t \tag{19}$$

Once again, after aggregating the variables, we get an economic system with six variables and six equations. The six equations are the resource constraint equation,

$$K_t = (1-\delta)K_{t-1} + (1-\tau)w_t H_t + r_t K_{t-1} - C_t + T_t \tag{20}$$

The first-order condition about labor,

$$\frac{u_h(C_t, H_t)}{u_c(C_t, H_t)} = -(1-\tau)w_t \tag{21}$$

And the Euler equation for consumption (15), the tax decision equation (18), and the wage and interest rate decision equations (12) and (13). Compared with the five-endogenous-variable C_t, H_t, K_t, w_t, r_t, five-equation economic system of the dispersed economy (12) to (16), the new system introduced a new endogenous variable and added a new tax decision equation (18), as well as the modified first-order condition about labor (21).

IV. CIA Model

Now, the RBC model can be expanded by introducing the cash-in-advance (CIA) model to include the role of money (Cooley and Hansen, 1989). This model posits that consumers must use cash to purchase goods, and the cash for consumption comes from the cash balance of the previous period and the transfer payment from the government. The government obtains the ability to make transfer payments by issuing money at a certain growth rate. Therefore, the individual's optimal plan can be written as,

$$Max \sum_{t=0}^{\infty} \beta^t u\left(c_t^i, h_t^i\right)$$
$$s.t. P_t c_t^i = m_{t-1}^i + (g_t - 1)M_{t-1}$$
$$c_t^i + k_t^i + \frac{m_t^i}{P_t} = w_t h_t^i + r_t k_{t-1}^i + (1-\delta)k_{t-1}^i + \frac{m_{t-1}^i + (g_t - 1)M_{t-1}}{P_t}$$

where $P_t$is the price level in period t. The first constraint is the cash constraint, meaning that consumption goods can only be purchased with cash, and cash comes from the cash balance of the previous period and the transfer payment from the government. Consistent with previous practices, lowercase letters m_t^i represent individual holdings of the previous period's cash balance, and uppercase letters M_t represent the total amount of money. The money growth in period t is $(g_t - 1)M_{t-1}$, where g_t is growth rate. Because individuals are homogeneous and distributed as a continuum on the interval [0,1], the mathematical form of the total transfer payment does not change when it is transferred to individuals on average. However, before obtaining the individual optimization conditions, this macro transfer payment cannot be algebraically operated with the individual's cash balance from the previous period. The second constraint is a modification of the classic resource constraint, also known as the flow constraint in the current model. On the right side are various income and stock variables from the previous period's balance: wage income, interest income, government transfer payment income, retained capital stock, and real cash balance; on the left side are the expenditure and stock variables for the current period's balance: consumption, current capital stock, and real cash balance.

Compared with the classic dispersed economic system (12) to (16), it is known that the CIA model introduces two new endogenous variables m_t^i and P_t[1] . Therefore, two new decision equations must be found. One is more intuitive, simply adding a first-order condition about m_t^i in the individual's optimal plan, and the other equation is the money growth equation $M_t = g_t M_{t-1}$. Specifically, the first constraint can be substituted into the second constraint to write the Lagrangian equation as follows,

$$L = \sum_{t=0}^{\infty} \beta^t u\left(c_t^i, h_t^i\right) + \sum_{t=0}^{\infty} \lambda_{1t} \left[c_t^i - \frac{m_{t-1}^i}{P_t} - \frac{(g_t - 1)M_{t-1}}{P_t}\right] + \sum_{t=0}^{\infty} \lambda_{2t} \left[k_t^i + \frac{m_t^i}{P_t} - w_t h_t^i - r_t k_{t-1}^i - (1-\delta)k_{t-1}^i\right]$$

The four first-order conditions about$c_t^i, h_t^i, k_t^i, m_t^i$ are,

$$\frac{\partial L}{\partial c_t^i} = \beta^t u_c(c_t^i, h_t^i) + \lambda_{1t} = 0$$

$$\frac{\partial L}{\partial h_t^i} = \beta^t u_h(c_t^i, h_t^i) - \lambda_{2t} w_t = 0$$

$$\frac{\partial L}{\partial k_t^i} = \lambda_{2t} - \lambda_{2,t+1}(r_t + 1 - \delta) = 0$$

$$\frac{\partial L}{\partial m_t^i} = \frac{\lambda_{2t}}{P_t} - \frac{\lambda_{1t+1}}{P_{t+1}} = 0$$

After eliminating$\lambda_{1t}, \lambda_{2t}$ and adding the two constraints, we have,

$$\frac{w_{t+1} \cdot u_h(c_t^i, h_t^i)}{w_t \cdot u_h(c_{t+1}^i, h_{t+1}^i) \cdot \beta} = r_t + 1 - \delta$$

$$\frac{u_h(c_t^i, h_t^i)}{w_t P_t} = -\frac{\beta u_c(c_{t+1}^i, h_{t+1}^i)}{P_{t+1}}$$

$$c_t^i - \frac{m_{t-1}^i}{P_t} - \frac{(g_t - 1)M_{t-1}}{P_t} = 0$$

$$k_t^i + \frac{m_t^i}{P_t} - w_t h_t^i - r_t k_{t-1}^i - (1-\delta)k_{t-1}^i = 0$$

Since the individuals are a continuum distributed in the interval [0,1], the individual variables are equal to the overall variables, i.e., $c_t^i = C_t, h_t^i = H_t, k_t^i = K_t, m_t^i = M_t$ and thus the above equation can be written as,

$$\frac{w_{t+1} \cdot u_h(C_t, H_t)}{w_t \cdot u_h(C_{t+1}, H_{t+1}) \cdot \beta} = r_t + 1 - \delta \tag{22}$$

$$\frac{u_h(C_t, H_t)}{w_t P_t} = -\frac{\beta u_c(C_{t+1}, H_{t+1})}{P_{t+1}} \tag{23}$$

$$C_t - \frac{M_{t-1}}{P_t} - \frac{(g_t - 1)M_{t-1}}{P_t} = 0 \tag{24}$$

$$K_t + \frac{M_t}{P_t} - w_t H_t - r_t K_{t-1} - (1-\delta)K_{t-1} = 0 \tag{25}$$

Compared with the three first-order conditions of the RBC model about c_t^i, h_t^i, k_t^i, the CIA model has an additional first-order condition about m_t^i. In addition, in the production sector's profit maximization, the determination of wages and interest rates can be obtained (i.e., equations (12) and (13)), plus the money supply decision rule, we have,

$$w_t = f_h(K_{t-1}, H_t)$$

$$r_t = f_k(K_{t-1}, H_t)$$

$$M_t = g_t M_{t-1} \tag{26}$$

A total of seven endogenous variables $C_t, H_t, K_t, M_t, P_t, w_t, r_t$, we have seven equations, which together constitute the rules of the CIA model's economic operation.

V. Imperfect Competition Model

On the basis of the CIA model, the wedge of imperfect competition can be introduced to make the model more in line with the real situation (Dixit and Stiglitz, 1977). This expansion mainly occurs in the production sector, and the consumption sector does not need to be changed. We divide the production sector into two stages: the production of final products and the production of intermediate products. The final product market is a perfectly competitive market, and the intermediate product market is a monopolistic competition market. It is assumed that the final product Y_t is produced by combining intermediate products in the following way,

$$Y_t = \left(\int_0^1 y_{jt}^{\frac{\xi-1}{\xi}} dj\right)^{\frac{\xi}{\xi-1}} \tag{27}$$

where y_{jt} is the j-th intermediate product in period t, and $\xi > 1$ is the substitution elasticity between different intermediate products. The above expression also means that intermediate products are a continuum distributed on the interval [0,1]. If the final product price is set to P_t and the intermediate product price is p_{jt}, then the final product manufacturer's profit maximization problem by selecting y_{jt} is,

$$\max P_t Y_t - \int_0^1 p_{jt} y_{jt} dj = P_t \left(\int_0^1 y_{jt}^{\frac{\xi-1}{\xi}} dj\right)^{\frac{\xi}{\xi-1}} - \int_0^1 p_{jt} y_{jt} dj$$

The first-order condition of this optimal plan, after rearrangement, is,

$$p_{jt} = P_t \left(\int_0^1 y_{jt}^{\frac{\xi-1}{\xi}} dj\right)^{\frac{\xi}{\xi-1}-1} y_{jt}^{\frac{\xi-1}{\xi}-1}$$

So, for two different intermediate products, we have,

$$\frac{p_{jt}}{p_{it}} = \left(\frac{y_{jt}}{y_{it}}\right)^{-\frac{1}{\xi}} \Longrightarrow p_{jt} = \left(\frac{y_{jt}}{y_{it}}\right)^{-\frac{1}{\xi}} p_{it}$$

Since the zero profit in the perfectly competitive market, we have,

$$P_t Y_t = \int_0^1 p_{jt} y_{jt} dj = \int_0^1 \left(\frac{y_{jt}}{y_{it}}\right)^{-\frac{1}{\xi}} p_{it} y_{jt} dj = p_{it} y_{it}^{\frac{1}{\xi}} \int_0^1 y_{jt}^{\frac{\xi-1}{\xi}} dj = p_{it} y_{it}^{\frac{1}{\xi}} Y_t^{\frac{\xi-1}{\xi}}$$

Thus, we have,

$$\frac{y_{jt}}{Y_t} = \left(\frac{p_{jt}}{P_t}\right)^{-\xi} \tag{28}$$

The above equation is the factor demand function for the final product. It should be noted that the zero-profit condition $P_t Y_t = \int_0^1 p_{jt} y_{jt} dj$ means that the price equals the marginal cost,

$$P_t = \int_0^1 \frac{p_{jt}y_{jt}}{Y_t} dj = \int_0^1 p_{jt}\left(\frac{p_{jt}}{P_t}\right)^{-\xi} dj$$

Thus, we have,

$$P_t = \left(\int_0^1 p_{jt}^{1-\xi} dj\right)^{\frac{1}{1-\xi}} \tag{29}$$

It can be seen that in the final product's optimal plan, we obtained the factor demand function (28), which introduces two new variables y_{jt}, p_{jt}, but we only have one equation, so we need to find a new equation, that is, how the price of intermediate products p_{jt} is determined.

Since the intermediate product market is monopolistic competition, solving its pricing rule is usually divided into two steps. The first step is to obtain the conditional factor demand function to obtain the cost function through cost minimization. The second step is to obtain the intermediate product pricing equation through profit maximization. The cost minimization problem can be written as,

$$\min w_t h_{jt} + r_t k_{j,t-1}$$

$$s.t.\, y_{jt} = A_t k_{jt}^{\theta} h_{j,t-1}^{1-\theta} \tag{30}$$

The constraint is the production function for intermediate goods and A_t is total factor productivity. The Lagrangian function can be written as,

$$L = w_t h_{jt} + r_t k_{j,t-1} - \lambda_t(y_{jt} - A_t k_{j,t-1}^{\theta} h_{jt}^{1-\theta})$$

Firms choose input factors $h_{jt}, k_{j,t-1}$ to minimize costs, noting that the Lagrange multiplier λ_t is the marginal cost[1]. The conditional factor demand function and marginal cost can be obtained as follows,

$$w_t = \frac{\lambda_t(1-\theta)y_{jt}}{h_{jt}} \tag{31}$$

$$r_t = \frac{\lambda_t \theta y_{jt}}{k_{j,t-1}} \tag{32}$$

$$\lambda_t = \frac{1}{A_t} \cdot \left(\frac{1}{\theta}\right)^{\theta} \left(\frac{1}{1-\theta}\right)^{1-\theta} r_t^{\theta} w_t^{1-\theta} \tag{33}$$

Then, looking at the profit maximization of intermediate products,

$$\max p_{jt}y_{jt} - P_t w_t h_{jt} - P_t r_t k_{j,t-1} \tag{34}$$

[1] The envelope theorem shows that the effect of the parameter (embodied here as y_{jt}) on the objective function (embodied here as cost) (embodied here as marginal cost) is equal to the value of the Lagrangian function's partial derivative of the parameter at the optimum, and the partial derivative of the Lagrangian function with respect to the parameter y_{jt} is λ_t.

$$s.t.\frac{y_{jt}}{Y_t} = \left(\frac{p_{Jt}}{P_t}\right)^{-\xi}$$

The constraint is the factor demand function of the final product, Eq. (28). Substituting the conditional factor demand function of the intermediate product Eq. (31) and Eq. (32) into the objective function and replacing y_{jt} using the constraints, and we have,

$$\begin{aligned} &\max p_{jt}y_{jt} - \lambda_t P_t(1-\theta)y_{jt} - \lambda_t P_t \theta y_{jt} \\ \Longrightarrow &\max p_{jt}y_{jt} - \lambda_t P_t y_{jt} \\ \Longrightarrow &\max p_{jt}\left(\frac{p_{jt}}{P_t}\right)^{-\xi} Y_t - \lambda_t P_t \left(\frac{p_{jt}}{P_t}\right)^{-\xi} Y_t \end{aligned}$$

Firms choose p_{jt} to maximize profits, and the first-order condition is,

$$p_{jt} = \frac{\xi}{\xi - 1}\lambda_t P_t \qquad (35)$$

So far, the model in the firm sector is solvable. Because there is a total of five endogenous variables $y_{jt}, p_{jt}, w_t, r_t, \lambda_t$ in the firm sector, and we have five equations (28), (31), (32), (33), and (35). In fact, we can obtain a more compact equation system through the aggregation processing of p_{jt}, y_{jt}. If intermediate products are homogeneous, equation (27) means $Y_t = y_{jt}$. Similarly, Eq. (29) means $p_{jt} = P_t$. Therefore, Eq. (35) indicates that the marginal cost is,

$$\lambda_t = \frac{\xi - 1}{\xi}$$

Based on Eq. (31), (32), and the intermediate product production function Eq. (30), the expressions for wages and interest rates can be obtained as,

$$w_t = \frac{\xi - 1}{\xi}(1-\theta)A_t\left(\frac{K_{t-1}}{H_t}\right)^{\theta} = \frac{\xi - 1}{\xi} f_h(K_{t-1}, H_t)$$

$$r_t = \frac{\xi - 1}{\xi}\theta A_t\left(\frac{K_{t-1}}{H_t}\right)^{\theta - 1} = \frac{\xi - 1}{\xi} f_k(K_{t-1}, H_t)$$

where the capital and labor on the interval [0,1] are continuum, which means $k_{jt} = K_t, h_{jt} = H_t$. Compared with the dispersed economy or CIA model's wage and interest rate determination equations (12) and (13), there is an additional markup coefficient $\frac{\xi-1}{\xi}$.

In addition, we can also obtain the aggregation of the monopoly profits of intermediate product manufacturers R_t. Based on the profit maximization Eq. (34), its aggregation on the interval [0,1], we have,

$$
\begin{aligned}
R_t &= \int_0^1 (p_{jt}y_{jt} - P_t w_t h_{jt} - P_t r_t k_{j,t-1})dj \\
&= P_t Y_t - P_t \int_0^1 \lambda_t (1-\theta) y_{jt} dj - P_t \int_0^1 \lambda_t \theta y_{jt} dj \\
&= P_t Y_t - P_t \lambda_t \int_0^1 y_{jt} dj
\end{aligned}
$$

The real profit is,

$$\frac{R_t}{P_t} = Y_t - \lambda_t \int_0^1 y_{jt} dj$$

The consumption sector's changes lie in the resource constraint condition (25). The income of consumers includes not only wage income and interest income but also the real profits $\frac{R_t}{P_t}$ brought by the monopoly of intermediate products (since enterprises are owned by consumers), that is,

$$K_t + \frac{M_t}{P_t} - w_t H_t - r_t K_{t-1} - \frac{R_t}{P_t} - (1-\delta)K_{t-1} = 0$$

But based on the income method of GDP accounting, we have,

$$Y_t = w_t H_t + r_t K_{t-1} + \frac{R_t}{P_t}$$

Therefore, the resource constraint can be rewritten as,

$$K_t + \frac{M_t}{P_t} - Y_t - (1-\delta)K_{t-1} = 0 \quad (36)$$

We can copy the CIA model's consumption sector's equations (22) to (24), plus the modified resource constraint equation (36). Because equation (36) introduces a new variable Y_t, we also need to add a new equation, that is, the aggregate production function. Finally, there is the money growth equation (26), the wage and interest rate determination equations, which together constitute the economic operation rules of the imperfect competition model.

$$
\begin{gathered}
\frac{w_{t+1} \cdot u_h(C_t, H_t)}{w_t \cdot u_h(C_{t+1}, H_{t+1}) \cdot \beta} = r_t + 1 - \delta \\
\frac{u_h(C_t, H_t)}{w_t P_t} = -\frac{\beta u_c(C_{t+1}, H_{t+1})}{P_{t+1}} \\
C_t - \frac{M_{t-1}}{P_t} - \frac{(g_t - 1)M_{t-1}}{P_t} = 0 \\
K_t + \frac{M_t}{P_t} - Y_t - (1-\delta)K_{t-1} = 0 \\
f(K_{t-1}, H_t) = Y_t = A_t K_{t-1}^{\theta} H_t^{1-\theta} \\
M_t = g_t M_{t-1} \\
w_t = \frac{\xi - 1}{\xi} f_h(K_{t-1}, H_t)
\end{gathered}
$$

$$r_t == \frac{\xi - 1}{\xi} f_k(K_{t-1}, H_t)$$

This model has a more endogenous variable Y_t than the CIA model, that is, it includes eight endogenous variables $C_t, H_t, K_t, Y_t, M_t, P_t, w_t, r_t$, and there are also eight equations. An important difference is that the determination rules for wages and interest rates have an additional markup coefficient $\frac{\xi-1}{\xi}$.

VI. New Keynesian Model

The New Keynesian model introduces price stickiness compared to the imperfect competition model (Gali, 2015). To facilitate the text, this paper only discusses price staggering, that is, price stickiness is reflected in the pricing of intermediate product manufacturers in each period t, not once in place, but only a portion of enterprises (ρ) can choose the optimal pricing p_t^*, and the rest maintain the previous period's price unchanged.

Therefore, compared with the imperfect competition model, we only need to modify the optimal plan of intermediate product profit maximization and the price index P_t that cannot be adjusted in one go to obtain the difference equation system of the New Keynesian model. Specifically, the profit maximization of intermediate products can be modified as,

$$\max \sum_{t=0}^{\infty} \beta^t \rho^t (p_t^* y_{jt} - P_t w_t h_{jt} - P_t r_t k_{j,t-1})$$

$$s.t. \frac{y_{jt}}{Y_t} = \left(\frac{p_t^*}{P_t}\right)^{-\xi} \quad (37)$$

The optimal price reflects that when intermediate product manufacturers can price optimally, they need to consider the probability ρ of their pricing continuing to the next period and the probability ρ^2 continuing to the period after that, etc. Since enterprises are family-owned, the appropriate discount factor for enterprises should be based on the ratio of the marginal utility of future consumption of the representative family to the marginal utility of current consumption, that is, β. The optimal plan can be further written as,

$$\max \sum_{t=0}^{\infty} \beta^t \rho^t \left(p_t^* y_{jt} - P_t TC_{jt}\right)$$
$$= \max \sum_{t=0}^{\infty} \beta^t \rho^t y_{jt} \left(p_t^* - P_t \lambda_{jt}\right)$$
$$= \max \sum_{t=0}^{\infty} \beta^t \rho^t \left(\frac{p_t^*}{P_t}\right)^{-\xi} Y_t \left(p_t^* - P_t \lambda_{jt}\right)$$

Where TC_{jt} is the total cost of the j enterprise. The first equality is because according to equations (31) and (32), substituting in $w_t h_{jt} + r_t k_{j,t-1}$, we get the total cost $TC_{jt} = \lambda_t y_{jt}$ and it can be seen that the average cost equals the marginal cost. The second equality is because the constraint condition (36) is substituted in. The first-order condition of this optimal plan, after a series of operations, can be obtained as,

$$p_t^* = \frac{\xi}{\xi - 1} \cdot \frac{\sum_{t=0}^{\infty} \beta^t \rho^t P_t^{\xi} Y_t \lambda_{jt}}{\sum_{t=0}^{\infty} \beta^t \rho^t P_t^{\xi-1} Y_t}$$

The optimal pricing includes the marginal cost, and the marginal cost is equation (33), that is,

$$\lambda_t = \frac{1}{A_t} \cdot \left(\frac{1}{\theta}\right)^{\theta} \left(\frac{1}{1-\theta}\right)^{1-\theta} r_t^{\theta} w_t^{1-\theta}$$

Since the marginal cost of all intermediate product enterprises is the same, we omit the subscript j. At the same time, also note that the original price summation equation (29), because only a ρ fraction of firms adjusts their prices in each period, changes that summation price equation to,

$$P_t^{1-\xi} = \rho P_{t-1}^{1-\xi} + (1-\rho)(p_t^*)^{1-\xi}$$

Different from the imperfect competition model, the current situation, the marginal cost cannot be simplified into $\frac{\xi-1}{\xi}$, so the determination equations for wages and interest rates need to be expressed in a different form. By dividing equations (31) and (32) and combining them with equation (30), we can obtain,

$$h_{jt} = \frac{y_{jt}}{A_t}\left[\frac{r_t(1-\theta)}{w_t\theta}\right]^{\theta}$$

$$k_{jt} = \frac{y_{jt}}{A_t}\left[\frac{r_t(1-\theta)}{w_t\theta}\right]^{\theta-1}$$

Summing up the labor and capital, and considering equation (28), we get,

$$H_t = \frac{1}{A_t}\left[\frac{r_t(1-\theta)}{w_t\theta}\right]^{\theta}\left[Y_t \int_0^1 \left(\frac{p_t^*}{P_t}\right)^{-\xi} dj\right]$$

$$K_t = \frac{1}{A_t}\left[\frac{r_t(1-\theta)}{w_t\theta}\right]^{\theta-1}\left[Y_t \int_0^1 \left(\frac{p_t^*}{P_t}\right)^{-\xi} dj\right]$$

The consumption sector is the same as the imperfect competition model. Therefore, the difference equation system of the New Keynesian model can be summarized as,

$$\frac{w_{t+1} \cdot u_h(C_t, H_t)}{w_t \cdot u_h(C_{t+1}, H_{t+1}) \cdot \beta} = r_t + 1 - \delta$$

$$\frac{u_h(C_t, H_t)}{w_t P_t} = -\frac{\beta u_c(C_{t+1}, H_{t+1})}{P_{t+1}}$$

$$C_t - \frac{M_{t-1}}{P_t} - \frac{(g_t - 1)M_{t-1}}{P_t} = 0$$

$$K_t + \frac{M_t}{P_t} - Y_t - (1-\delta)K_{t-1} = 0$$

$$M_t = g_t M_{t-1}$$

$$H_t = \frac{1}{A_t}\left[\frac{r_t(1-\theta)}{w_t\theta}\right]^{\theta}\left[Y_t \int_0^1 \left(\frac{p_t^*}{P_t}\right)^{-\xi} dj\right]$$

$$K_t = \frac{1}{A_t}\left[\frac{r_t(1-\theta)}{w_t\theta}\right]^{\theta-1}\left[Y_t \int_0^1 \left(\frac{p_t^*}{P_t}\right)^{-\xi} dj\right]$$

$$p_t^* = \frac{\xi}{\xi-1}\cdot\frac{\sum_{t=0}^{\infty}\beta^t\rho^t P_t^{\xi} Y_t \lambda_t}{\sum_{t=0}^{\infty}\beta^t\rho^t P_t^{\xi-1} Y_t}$$

$$\lambda_t = \frac{1}{A_t}\cdot\left(\frac{1}{\theta}\right)^{\theta}\left(\frac{1}{1-\theta}\right)^{1-\theta} r_t^{\theta} w_t^{1-\theta}$$

$$P_t^{1-\xi} = \rho P_{t-1}^{1-\xi} + (1-\rho)(p_t^*)^{1-\xi}$$

This is a New Keynesian model with ten endogenous variables $C_t, H_t, K_t, M_t, P_t, p_t^*, w_t, r_t, \lambda_t, Y_t$ and ten equations. Unlike the imperfect competition model, the above equation system does not include the aggregate production function because we have added the marginal cost equation, and equations (30), (31), (32), and (33) are only three independent equations.

VII. Conclusion

This paper uses "solving equations" as a clue to explain how to start from the simple Solow model, gradually introduce new endogenous variables, add new equations, and obtain the RBC model, CIA model, imperfect competition model, and the New Keynesian model. It can be seen that the form of introducing new equations when adding new endogenous variables is diverse. It can be like the expansion from the Solow model to the RBC model, introducing consumer optimal planning to obtain the first-order condition type equation. It can also be like the internalization of labor in the RBC model, adding new first-order conditions under the same optimal plan, and it can even be like the direct addition of government budget balance equations when introducing government departments in the RBC model. For students, the key is to master the form of adding new endogenous variables and introducing new equations from the above expansion, and in their future research, according to research needs, expand existing models to form new model designs, providing clear and rigorous theoretical logic for empirical research.

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Reference

1. Ascher U M, Mattheij R M M, Russell R D. Numerical Solution of Boundary Value Problems for Ordinary Differential Equations [M]. Philadelphia: SIAM, 1995.
2. Benassy-Quere A, Coeure B, Jacquet P, et al. Economic Policy: Theory and Practice[M]. New York: Oxford University Press, 2010.
3. Blanchard O, Kahn C. The Solution of Linear Difference Models under Rational Expectations [J]. Ecnometrica, 1980, 48(5): 1305-1311.
4. Blanchard O J, Fischer S. Lectures on Macroeconomics[M]. The MIT Press, 1989.
5. Cooley T F, Hansen G D. The Inflation Tax in a Real Business Cycle Model[J]. The American Economic Review, 1989, 79(4): 73 3-748.
6. Cunningham S. Causal Inference: The Mixtape[M]. Yale University Press, 2021.
7. DeJong D N, Dave C. Structural Macroeconometrics[M]. 2nd ed. Princeton: Princeton University Press, 2011.
8. Dixit A, Stiglitz J. Monopolistic Competition and Optimum Product Diversity[J]. American Economic Review, 1977, 67(3): 297-308.
9. Gali J. Monetary Policy, Inflation, and the Business Cycle[M]. 2nd ed. Princeton: Princeton University Press, 2015.

10. Hansen G. Indivisible Labor and the Business Cycle [J]. Journal of Monetary Economics, 19 85, 16: 309-328.
11. Heer B, Maussner A. Dynamic General Equilibrium Modeling[M]. Berlin: Springer, 2024.
12. Heijdra B J. Foundations of Modern Macroeconomics[M]. 3rd ed. Oxford University Press, 2017.
13. McCandless G. The ABCs of RBCs[M]. Harvard University Press, 2008.
14. Romer D. Advanced Macroeconomics[M]. 5 e. Mcgraw-Hill, 2017.
15. Solow R M. A Contribution to the Theory of Economic Growth[J]. Quarterly Journal of Economics, 1956, 70: 65-94.
16. Uhlig H. A Toolkit for Analysising Nonlinear Dynamic Stochastic Models Easily [M] // Marimon R, Scott A. Computational Methods for the Study of Dynamic Economies. Oxford: Oxford University Press, 1999: 30-61.