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Original Article

Three-Parameter Generalized Quaternions Based on Gaussian-Leonardo Numbers

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Abstract:
This paper aims to extend the theory of quaternions by introducing a novel type of 3-parameter generalized quaternion, whose components are Gaussian Leonardo (abbreviated as G-L) numbers, referred to as the G-L 3-parameter generalized quaternions. Through systematic investigation of their recurrence relations, Binet-like formulas, generating functions, exponential generating functions, Poisson generating functions, as well as several important identities and summation formulas, we not only uncover the rich mathematical structure of these quaternions but also provide new tools for applications in related fields. Additionally, by utilizing the S-matrix and representation matrix of the G-L 3-parameter generalized quaternions, a new identity is discovered, further perfecting the theoretical framework. These results offer significant references for the deepening and expansion of quaternion theory.
Keywords: Gaussian Leonardo numbers; 3-parameter generalized quaternions; Recurrence relations; Generating functions; Matrix representations
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1 Introduction

Number systems have important applications in various disciplines and have therefore become a focus of research. Quaternions, as an important numerical system, have attracted widespread attention since Hamilton introduced them in 1843 as an extension of complex numbers. Quaternion algebra is a non-commutative, associative four-dimensional Clifford algebra with significant applications in graph theory, computer science, differential geometry, physics, and other fields.

The set of real quaternions [1, 2, 3] can be expressed as:

$$H = \{q|q = q_0 + q_1 i + q_2 j + q_3 k, q_0, q_1, q_2, q_3 \in R\}$$

where i,j,k are quaternion units satisfying the multiplication rules i2 = j2 = k2 = −1, ij = −ji = k, jk = −kj = i, ki = −ik = j. Building on the study of real quaternions, James Cockle further investigated split quaternions [4]. The split quaternion units satisfy the multiplication rules i2 = −1, j2 = k2 = 1, ij = −ji = k, jk = −kj = −i, ki = −ik = j. Additionally, many studies have explored generalized quaternions (or 2-parameter generalized quaternions, abbreviated as 2PGQ). The set of 2-parameter generalized quaternions is denoted by Hλ1,λ2 and defined as:

$$H\lambda_{1,\lambda 2} = \{q|q = q_0 + q_1 i + q_2 j + q_3 k, q_0, q_1, q_2, q_3, \lambda_1, \lambda_2 \in R\},$$

where the quaternion units i,j,k satisfy the multiplication rules i2 = −λ1, j2 = −λ2 , k2 = −λ1λ2 , ij = −ji = k , jk = −kj = λ2i , ki = −ik = λ1j.Specifically, when λ1 = λ2 = 1, q is a real quaternion; when λ1 = 1,λ2 = −1, q is a split quaternion; when λ1 = 1,λ2 = 0, q is a semi-quaternion; when λ1 = −1,λ2 = 0, q is a split semi-quaternion; and when λ1 = λ2 = 0, q is a 1/4-quaternion. For more details, see [5, 6, 7].

Building on this foundation, T. D. Sentu¨rk and Z. Unal [10] introduced a new family of quaternions—3-parameter generalized quaternions (abbreviated as 3PGQs). These quaternions generalize real quaternions, split quaternions, and 2-parameter generalized quaternions based on three parameters, providing a comprehensive interpretation of quaternion algebra. The set of 3-parameter generalized quaternions is denoted by K and defined as:

$K = \{q | q = q_0 + q_1 i + q_2 j + q_3 k , q_0 , q_1 , q_2 , q_3 , \lambda_1 , \lambda_2 , \lambda_3 \in R\}$,

where the quaternion units i, j, k satisfy the multiplication rules given in Table 1.

Table 1: Multiplication rules for 3-parameter generalized quaternion units

	i	j	k
i	$-\lambda_1$	k	$-\lambda_2 j$
j	$-k$	$-\lambda 1 \lambda 3$	$\lambda_2 i$
k	$\lambda_2 j$	$-\lambda_3 i$	$-\lambda 2 \lambda 3$

Depending on the values of $\lambda_i \in \{1,2,3\}$, when $\lambda_1 = 1$, $\lambda_2, \lambda_3 \in R$, q is a 2PGQ [5]; when $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = -1$, q is a split quaternion [10]; when $\lambda_1 = \lambda_2 = \lambda_3 = 1$, q is a real quaternion [6]; when $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = 0$, q is a semi-quaternion [8]; when $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 0$, q is a split semiquaternion [9]; and when $\lambda_1 = 1, \lambda_2 = \lambda_3 = 0$, q is a 1/4-quaternion [9].

On the other hand, integer sequences and their applications have always been a central theme in number theory research. Among them, the Gaussian Leonardo sequence (abbreviated as G-L) has garnered significant attention. It is defined by the recurrence relation $GL_n = Le_n + Le_{n-1} i$, $n \geq 1$, 3. where GL_n denotes the n-th G-L number and Le_n denotes the n-th Leonardo number [11]. According to [12], the G-L sequence has the following algebraic properties:

1. The recurrence relation of the G-L sequence can be written as:

$GL_n = 2GL_{n-1} - GL_{n-3}, n \geq 3$,

with initial values $GL_0 = 1 - i$, $GL_1 = 1 + i$, $GL_2 = 3 + i$.

2. The generating function of the G-L sequence is:

$$\sum_{n=0}^{\infty} GL_n x^n = \frac{(1-i) + (-1+3i)x + (1-i)x^2}{1 - 2x + x^3}.$$

4. The Binet formula for the G-L sequence is:
5.

$$GL_n = A\alpha^n + B\beta^n + C \cdot 1^n,$$

where $A = 1 + \frac{1}{\sqrt{5}} + \frac{2}{\sqrt{5}} i, B = 1 - \frac{1}{\sqrt{5}} - \frac{2}{\sqrt{5}} i, C = -1 - i, and \ \alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}$

are the roots of the characteristic equation $x^3 - 2x^2 + 1 = 0$ other than 1. Clearly, $\alpha + \beta = 1, \alpha - \beta = \sqrt{5}, \alpha\beta = -2.$

In recent years, the direction of combining different types of quaternions with special recurrence sequences has attracted considerable attention from scholars. For example, [13] pioneered the study of Fibonacci quaternions and their recurrence relations; [14] and [15] introduced the concepts of generalized Fibonacci quaternions and split k-Fibonacci and split k-Lucas quaternions, respectively; [16, 17, 18, 19] discussed Jacobsthal quaternions, Pell quaternions, Pell-Lucas quaternions, $h(x)$-Fibonacci quaternion polynomials, and modified Pell quaternions. These studies further expanded the scope of quaternion research. Additionally, researchers have conducted in-depth investigations into the split, mixed, dual, and higher-order generalizations of quaternions, providing new insights for quaternion studies.

Inspired by the above literature, this paper introduces a new type of 3-parameter generalized quaternion whose components are G-L numbers, referred to as G-L 3-parameter generalized quaternions. We explore several of their properties, including recurrence relations, Binet-like formulas, generating functions, exponential generating functions, Poisson generating functions, important identities, and summation formulas. Finally, we define the S-matrix and representation matrix for G-L 3-parameter generalized quaternions and derive a new identity.

G-L 3-Parameter Generalized Quaternions

The core objective of this section is to construct a 3-parameter generalized quaternion system based on Gaussian Leonardo numbers and present some elegant results.

Definition 1. Let QGL_n be the n-th G-L 3-parameter generalized quaternion, defined as:

$$QGL_n = GL_n + iGL_{n+1} + jGL_{n+2} + kGL_{n+3}, \qquad n \geq 3,$$

where GL_n is the n-th G-L number, and i, j, k satisfy the multiplication rules given in Table 1.

Next, we study some algebraic operations of G-L 3-parameter generalized quaternions, such as equality, addition and subtraction, scalar multiplication, scalar and vector parts, multiplication, conjugate, norm, inverse, and inner product.

For $\forall n,m \in \mathrm{N} \cup \{0\}$, let $QGL_n = GL_n + iGL_{n+1} + jGL_{n+2} + kGL_{n+3}$ and $QGL_m = GL_m + iGL_{m+1} + jGL_{m+2} + kGL_{m+3}$ be the n-th and m-th G-L 3-parameter generalized quaternions, respectively. Then:

1. Equality:

$$QGL_n = QGL_m \Leftrightarrow GL_n = GL_m, \qquad GL_{n+1} = GL_{m+1},$$
$$GL_n+2 = GL_m+2, \qquad GL_n+3 = GL_m+3.$$

2. Addition/Subtraction:

$$QGL_n \pm QGL_m = (GL_n \pm GL_m) + i(GL_{n+1} \pm GL_{m+1})$$
$$+ j(GLn+2 \pm GLm+2) + k(GLn+3 \pm GLm+3).$$

3. Scalar Multiplication:

$$cQGL_n = cGL_n + ciGL_{n+1} + cjGL_{n+2} + ckGL_{n+3}, c \in \mathrm{R}.$$

4. Scalar and Vector Parts: The scalar and vector parts of QGL_n are denoted by S_{QGLn} and V_{QGLn}, respectively, where $S_{QGLn} = GL_n$ and $V_{QGLn} = iGL_{n+1} + jGL_{n+2} + kGL_{n+3}$. This implies:

$$SQGL_n{\pm}QGL_m = GLn \pm GLm = SQGL_n \pm SQGL_m, VQGL_n{\pm}QGL_m = VQGL_n \pm VQGL_m.$$

5. Multiplication:

$$QGL_n \cdot QGL_m = SQGL_n \cdot SQGL_m - f(VQGL_n, VQGL_m)$$
$$+ SQGL_n \cdot VQGL_m + SQGL_m \cdot VQGL_n + VQGL_n \wedge VQGL_m,$$

Where

$$f(VQGL_n, VQGL_m) = \lambda 1\lambda 2GLn+1GLm+1 + \lambda_1\lambda_3 GL_n+2GL_m+2 + \lambda_2\lambda_3 GL_n+3GL_m+3,$$

and

$$V_{QGL_n} \wedge V_{QGL_m} = \begin{vmatrix} i & j & k \\ GL_{n+1} & GL_{n+2} & GL_{n+3} \\ GL_{m+1} & GL_{m+2} & GL_{m+3} \end{vmatrix}.$$

6. Conjugate:

$$QGL_n = GL_n - iGL_{n+1} - jGL_{n+2} - kGL_{n+3} \ .$$

7. Inverse:

$$QGL_n^{-1} = \overline{QGL_n}/N_{QGL_n},$$

where $N_{QGL_n} = GL_n^2 + \lambda_1\lambda_2 GL_{n+1}^2 + \lambda_1\lambda_3 GL_{n+2}^2 + \lambda_2\lambda_3 GL_{n+3}^2 \neq 0$.

8. Inner Product:

$$\langle QGL_n, QGL_m\rangle = GL_nGL_m + \lambda_1\lambda_2 GL_{n+1}GL_{m+1} + \lambda_1\lambda_3 GL_{n+2}GL_{m+2} + \lambda_2\lambda_3 GL_{n+3}GL_{m+3} \ .$$

9. Norm:

$$N_{QGL_n} = \langle QGL_n, QGL_n\rangle = GL_n^2 + \lambda_1\lambda_2 GL_{n+1}^2 + \lambda_1\lambda_3 GL_{n+2}^2 + \lambda_2\lambda_3 GL_{n+3}^2 \ .$$

Thus, the G-L 3-parameter generalized quaternions form a commutative group under addition and a four-dimensional non-commutative associative algebra over the real numbers, constituting a division ring.

Using the definition of GL_n in (1), we can write:

$$\begin{aligned} QGL_n &= GL_n + iGL_{n+1} + jGL_{n+2} + kGL_{n+3} \\ &= (2GL_{n-1} - GL_{n-3}) + i(2GL_n - GL_{n-2}) \\ &+ j(2GL_{n+1} - GL_{n-1}) + k(2GL_{n+2} - GL_n), \end{aligned}$$

$$QGL_{n-1} = GL_{n-1} + iGL_n + jGL_{n+1} + kGL_{n+2} \ ,$$

$$QGL_{n-3} = GL_{n-3} + iGL_{n-2} + jGL_{n-1} + kGL_n \ .$$

Thus, we have:

Theorem 1. (Recurrence Formula) For $n \geq 3$, the recurrence formula for G-L 3-parameter generalized quaternions is:

$$QGL_n = 2QGL_{n-1} - QGL_{n-3} \ ,$$

with initial values $QGL_0 = 3(1-\lambda_2)j+(5+\lambda_1)k$, $QGL_1 = 4i+5(1-\lambda_2)j+ 3(3 + \lambda_1)k$, $QGL_2 = 6i + 9(1 - \lambda_2)j + 5(3 + \lambda_1)k$.

To find the generating function of G-L 3-parameter generalized quaternions, we express the sequence as a power series where each term corresponds to a coefficient.

Theorem 2. (Generating Function) The generating function for the n-th G-L 3-parameter generalized quaternion QGL_n is:

$$g(x) = \sum_{n=0}^{\infty} QGL_n x^n = \frac{E + Fx + Gx^2}{1 - 2x + x^3},$$

where $E = 3(1 - \lambda_2)j + (5 + \lambda_1)k$, $F = 4i + (\lambda_2 - 1)j + (\lambda_1 - 1)k$, $G = -2i + (\lambda_2 - 1)j - (\lambda_1 + 3)k$.

Proof. Since the generating function $g(x)$ can be expressed as a formal power series:

$$g(x) = QGL_0 + QGL_1x + QGL_2x^2 + \cdots + QGL_nx^n + \cdots \ ,$$

$$2xg(x) = 2QGL_0x + 2QGL_1x^2 + 2QGL_2x^3 + \cdots + 2QGL_{n-1}x^n + \cdots ,$$

$$-x^3g(x) = -QGL_0x^3 - QGL_1x^4 - QGL_2x^5 - \cdots - QGL_{n-3}x^n - \cdots ,$$

we have:

$$(1-2x+x^3)g(x) = QGL_0+(QGL_1-2QGL_0)x+(QGL_2-2QGL_1)x^2+\cdots .$$

Thus:

$$g(x) = \frac{QGL_0 + (QGL_1 - 2QGL_0)x + (QGL_2 - 2QGL_1)x^2}{1 - 2x + x^3} = \frac{E + Fx + Gx^2}{1 - 2x + x^3} .$$

Theorem 3. (Binet-like Formula) The Binet-like formula for the n-th G-L 3-parameter generalized quaternion QGL_n is:

$$\mathrm{QGL}_n = A\hat{\alpha}\alpha^n + B\hat{\beta}\beta^n + C\hat{1}\cdot 1^n,$$

where A,B,C are as in (3), $\hat{\alpha} = 1+\alpha i+\alpha^2 j+\alpha^3 k, \hat{\beta} = 1+\beta i+\beta^2 j+\beta^3 k,$

and $\hat{1} = 1+i+j+k.$

Proof. Using the Binet formula (3) for G-L numbers and the definition of G-L 3-parameter generalized quaternions, we have:

$$QGLn = GLn + GLn{+}1i + GLn{+}2j + GLn{+}3k$$

$$= A\alpha^n(1 + \alpha i + \alpha^2 j + \alpha^3 k)+ B\beta^n(1 + \beta i + \beta^2 j + \beta^3 k)+ C(1 + i + j + k)$$

$$= A\hat{\alpha}\alpha^n + B\hat{\beta}\beta^n + C\hat{1}\cdot 1^n.$$

Theorem 4. (Exponential Generating Function) The exponential generating function for the n-th G-L 3-parameter generalized quaternion QGL_n is:

$$h(x) = \sum_{n=0}^{\infty} QGL_n \frac{x^n}{n!} = A\hat{\alpha}e^{\alpha x} + B\hat{\beta}e^{\beta x} + C\hat{1}e^{x} .$$

Proof. Using equation (7), we have:

$$h(x) = A\hat{\alpha}\sum_{n=0}^{\infty}\frac{(\alpha x)^n}{n!} + B\hat{\beta}\sum_{n=0}^{\infty}\frac{(\beta x)^n}{n!} + C\hat{1}\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$$= A\hat{\alpha}e^{\alpha x} + B\hat{\beta}e^{\beta x} + C\hat{1}e^{x}.$$

Theorem 5. (Poisson Generating Function) The Poisson generating function for the n-th G-L 3-parameter generalized quaternion QGL_n is:

$$p(x) = e^{-x}\sum_{n=0}^{\infty} QGL_n \frac{x^n}{n!} = A\hat{\alpha}e^{(\alpha-1)x} + B\hat{\beta}e^{(\beta-1)x} + C\hat{1} .$$

Proof. This follows immediately from Theorem 4.

Theorem 6.[Vajda's Identity] Let m, n, k be non-negative integers. Then the following identity holds for G-L 3-parameter generalized quaternions:

$$\mathrm{QGL}_{n+m}\cdot \mathrm{QGL}_{n+k} - \mathrm{QGL}_n \cdot \mathrm{QGL}_{n+m+k} = AB(-2)^n(\alpha^m - \beta^m)\left(\hat{\alpha}\hat{\beta}\beta^k - \hat{\beta}\hat{\alpha}\alpha^k\right)$$

$$+ AC\alpha^n(\alpha^m - 1)\left(\hat{\alpha}\hat{1} - \hat{1}\alpha^k\right) + BC\beta^n(\beta^m - 1)\left(\hat{\beta}\hat{1} - \hat{1}\beta^k\right),$$

where A,B,C are constants defined in equation (3), with $\hat{\alpha} = 1 + \alpha i + \alpha^2 j + \alpha^3 k$,

$$\hat{\beta} = 1 + \beta i + \beta^2 j + \beta^3 k, \text{ and } \hat{1} = 1 + i + j + k.$$

Proof Using the Binet-like formula for G-L 3-parameter generalized quaternions from equation (7):

$$QGL_n = A\hat{\alpha}\alpha^n + B\hat{\beta}\beta^n + C\hat{1},$$

we expand both sides of the identity:

$$\begin{aligned} &QGL_{n+m} \cdot QGL_{n+k} - QGL_n \cdot QGL_{n+m+k} \\ &= \left(A\hat{\alpha}\alpha^{n+m} + B\hat{\beta}\beta^{n+m} + C\hat{1}\right) \cdot \left(A\hat{\alpha}\alpha^{n+k} + B\hat{\beta}\beta^{n+k} + C\hat{1}\right) \\ &\quad - \left(A\hat{\alpha}\alpha^{n} + B\hat{\beta}\beta^{n} + C\hat{1}\right) \cdot \left(A\hat{\alpha}\alpha^{n+m+k} + B\hat{\beta}\beta^{n+m+k} + C\hat{1}\right) \end{aligned}$$

Expanding the products and canceling terms using the linearity of quaternion multiplication:

$$= \mathrm{AB}(\alpha\beta)^{\mathrm{n}} \left[\alpha^m \beta^k \hat{\alpha}\hat{\beta} - \beta^m \alpha^k \hat{\beta}\hat{\alpha}\right]$$
$$+ \mathrm{AC}\alpha^{\mathrm{n}} \left[\alpha^m \hat{\alpha}\hat{1} - \hat{1}\alpha^{m+k}\right] + \mathrm{BC}\beta^{\mathrm{n}} \left[\beta^m \hat{\beta}\hat{1} - \hat{1}\beta^{m+k}\right]$$

Simplifying each component using the characteristic equation $\alpha\beta = -2$:

$$AB(\alpha\beta)^n = AB(-2)^n$$
$$\alpha^m - \beta^m = (\alpha^m - \beta^m)$$
$$\alpha^k - \beta^k = (\alpha^k - \beta^k)$$

Substituting these results back into the expression:

$$= \mathrm{AB}(-2)^{\mathrm{n}} \left(\alpha^{\mathrm{m}} - \beta^{\mathrm{m}}\right)\left(\hat{\alpha}\hat{\beta}\beta^k - \hat{\beta}\hat{\alpha}\alpha^k\right)$$
$$+ \mathrm{AC}\alpha^{\mathrm{n}}\left(\alpha^m - 1\right)\left(\hat{\alpha}\hat{1} - \hat{1}\hat{\alpha}\right) + \mathrm{BC}\beta^{\mathrm{n}}\left(\beta^m - 1\right)\left(\hat{\beta}\hat{1} - \hat{1}\hat{\beta}\right)$$

This completes the proof of Vajda's identity for G-L 3-parameter generalized quaternions.

Corollary 1. If we set $m = -k$ in (10), we obtain the Catalan identity for G-L 3-parameter generalized quaternions:

$$QGL_{n-k} \cdot QGL_{n+k} - (QGL_n)^2 = AB(-2)^{n-k}\left(\beta^k - \alpha^k\right)\left(\hat{\alpha}\hat{\beta}\beta^k - \hat{\beta}\hat{\alpha}\alpha^k\right)$$
$$+ AC\alpha^{n-k}\left(1 - \alpha^k\right)\left(\hat{\alpha}\hat{1} - \hat{\alpha}\alpha^k\right) + BC\beta^{n-k}\left(1 - \beta^k\right)\left(\hat{\beta}\hat{1} - \hat{\beta}\beta^k\right).$$

Corollary 2. If we set $k = -m = 1$ in (10), we obtain the Cassini identity for G-L 3-parameter generalized quaternions:

$$\begin{aligned} QGL_{n-1} \cdot QGL_{n+1} - (QGL_n)^2 = (-1)^{n+1} AB\sqrt{5} \cdot 2^{n-1}(\hat{\alpha}\hat{\beta}\beta - \hat{\beta}\hat{\alpha}\alpha) \\ + AC\alpha^{n-1}\beta(\hat{\alpha}\hat{1} - \hat{\alpha}\alpha) \\ + BC\beta^{n-1}\alpha(\hat{\beta}\hat{1} - \hat{\beta}\beta). \end{aligned}$$

Corollary 3. If we set $k = r - n$ and $m = 1$ in (10), we obtain the d'Ocagne identity for G-L 3-parameter generalized quaternions:

$$\begin{aligned}QGL_{n+1}\cdot QGL_r - QGL_n\cdot QGL_{r+1} &= (-1)^n AB\sqrt{5}\cdot 2^n\left(\hat{\alpha}\hat{\beta}\beta^{r-n} - \hat{\beta}\hat{\alpha}\alpha^{r-n}\right)\\ &\quad + 2AC\alpha^{n-1}\left(\hat{\alpha}\hat{1} - \hat{\alpha}\alpha^{r-n}\right)\\ &\quad + 2BC\beta^{n-1}\left(\hat{\beta}\hat{1} - \hat{\beta}\beta^{r-n}\right).\end{aligned}$$

Next, using results from [20], we present some summation formulas for G-L 3-parameter generalized quaternions.

Theorem 7. (Summation Formulas) For $n \geq 0$, the following summation formulas hold for G-L 3-parameter generalized quaternions:

(i)

$$\sum_{k=0}^{n} QGL_k = QGL_{n+2} - n\omega_1 - \delta_1 \ ,$$

where $\omega_1 = 1 - \lambda_1\lambda_2 + 2i + (1 + \lambda_2)j + (1 - \lambda_1)k$, $\delta_1 = 2(1 - \lambda_1\lambda_2) + 6i + 6(1 + \lambda_2)j + 2(5 - 2\lambda_1)k$.

(ii)

$$\sum_{k=0}^{n} QGL_{2k} = QGL_{2n+2} - 2n\omega_2 - \delta_2,$$

where $\omega_2 = 1-\lambda_1\lambda_2 + 2i + (1+\lambda_2)j + (1-\lambda_1)k$, $\delta_2 = 2(1-2\lambda_1\lambda_2) + 4i + (2 + 4\lambda_2)j - 4\lambda_1 k$.

(iii)

$$\sum_{k=0}^{n} QGL_{k+1} = QGL_{n+3} - 6n\omega_3 - \delta_3,$$

where $\omega_3 = 1 - i$, $\delta_3 = 12 + 14i + (5 + 4\lambda_2)j + (9 - \lambda_1)k$.

Proof.

(i) Using the definition of QGL_n in (4) and Theorem 2.6 from [22], we derive:

$$\begin{aligned}\sum_{k=0}^{n} QGL_k &= \sum_{k=0}^{n}(GL_k + iGL_{k+1} + jGL_{k+2} + kGL_{k+3})\\ &= \sum_{k=0}^{n} GL_k + i\sum_{k=0}^{n} GL_{k+1} + j\sum_{k=0}^{n} GL_{k+2} + k\sum_{k=0}^{n} GL_{k+3}\\ &= (GL_{n+2} - (n+2)(1+i)) + i\left(GL_{n+3} - (n+3)(1+i) - GL_0\right)\\ &\quad + j\left(GL_{n+4} - (n+4)(1+i) - GL_0 - GL_1\right)\\ &\quad + k\left(GL_{n+5} - (n+5)(1+i) - GL_0 - GL_1 - GL_2\right)\\ &= (GL_{n+2} + iGL_{n+3} + jGL_{n+4} + kGL_{n+5})\\ &\quad - [(n+2) + (n+3)i + (n+4)j + (n+5)k](1+i)\\ &\quad - (i+j+k)GL_0 - (j+k)GL_1 - kGL_2\\ &= QGL_{n+2} - n\omega_1 - \delta_1,\end{aligned}$$

where $\omega_1 = 1 - \lambda_1\lambda_2 + 2i + (1 + \lambda_2)j + (1 - \lambda_1)k$, $\delta_1 = 2(1 - \lambda_1\lambda_2) + 5i + (2 + \lambda_2)j + (9 - 4\lambda_1)k$.

(ii) Using Theorem 2.6 and Theorem 2.7 from [22], we obtain:

$$\sum_{k=0}^{n} QGL_{2k} = \sum_{k=0}^{n} (GL_{2k} + iGL_{2k+1} + jGL_{2k+2} + kGL_{2k+3})$$
$$= \sum_{k=0}^{n} GL_{2k} + i\sum_{k=0}^{n} GL_{2k+1} + j\sum_{k=0}^{n} GL_{2k+2} + k\sum_{k=0}^{n} GL_{2k+3}$$
$$= (GL_{2n+2} - (2n+2)(1+i)) + i\sum_{i=1}^{2n+1} GL_i + j\sum_{i=2}^{2n+2} GL_i + k\sum_{i=3}^{2n+3} GL_i$$
$$= GL_{2n+2} - (2n+2)(1+i) + i\,(GL_{2n+3} - (2n+3)(1+i) - GL_0)$$
$$+ j\,(GL_{2n+4} - (2n+4)(1+i) - GL_0 - GL_1)$$
$$+ k\,(GL_{2n+5} - (2n+5)(1+i) - GL_0 - GL_1)$$
$$= (GL_{2n+1} + iGL_{2n+3} + jGL_{2n+4} + kGL_{2n+5})$$
$$- [(2n+2) - \lambda_1\lambda_2(2n+3) + (4n+5)i + (2n(1+\lambda_2) + 5\lambda_2 + 4)j$$
$$+(2n(1-\lambda_1) - 4\lambda_1 + 5)k] - [\lambda_1\lambda_2 + i + (2+\lambda_2)j + 5k]$$
$$= QGL_{2n+2} - \omega_2 - \delta_2,$$

where $\omega_2 = 1-\lambda_1\lambda_2 + 2i + (1+\lambda_2)j + (1-\lambda_1)k$, $\delta_2 = 2(1-2\lambda_1\lambda_2) + 4i + (2+4\lambda_2)j - 4\lambda_1 k$.

(iii) The proof is similar to the above, utilizing Theorem 2.6, Theorem2.7, and Theorem 2.8 from [22], and is omitted here due to space limitations.

Matrix Representation of G-L 3-Parameter Generalized Quaternions

It is well known that one of the most useful techniques for generating sequences is the matrix method. This section presents the representation matrix for G-L 3-parameter generalized quaternions.

Using equation (5) and mathematical induction, we obtain:

$$\begin{bmatrix} QGL_{n+2} \\ QGL_{n+1} \\ QGL_n \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} QGL_2 \\ QGL_1 \\ QGL_0 \end{bmatrix}.$$

Definition 2. The matrices:

$$S = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

is called the S-matrix.

Theorem 8.[Matrix Power Representation] For all $n \geq 0$, the n-th power of the S-matrix satisfies:

$$S^n M = \begin{bmatrix} QGL_{n+3} & QGL_{n+2} & QGL_{n+1} \\ QGL_{n+2} & QGL_{n+1} & QGL_n \\ QGL_{n+1} & QGL_n & QGL_{n-1} \end{bmatrix},$$

where M is the initial matrix:

$$M = \begin{bmatrix} QGL_3 & QGL_2 & QGL_1 \\ QGL_2 & QGL_1 & QGL_0 \\ QGL_1 & QGL_0 & QGL_{-1} \end{bmatrix}.$$

Proof. We prove this by mathematical induction.

Base Case ($n = 1$): Direct computation using matrix multiplication:

$$\text{SM} = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} QGL_3 & QGL_2 & QGL_1 \\ QGL_2 & QGL_1 & QGL_0 \\ QGL_1 & QGL_0 & QGL_{-1} \end{bmatrix}$$

$$= \begin{bmatrix} 2QGL_3 - QGL_1 & QGL_2 - QGL_0 & QGL_1 - QGL_{-1} \\ QGL_3 & QGL_2 & QGL_1 \\ QGL_2 & QGL_1 & QGL_0 \end{bmatrix}$$

Using the recurrence relation $QGL_n = 2QGL_{n-1} - QGL_{n-3}$, we verify:

$$2QGL_3 - QGL_1 = QGL_4\,,$$

$$2QGL_2 - QGL_0 = QGL_3\,,$$

$$2QGL_1 - QGL_{-1} = QGL_2\,.$$

Thus:

$$\text{SM} = \begin{bmatrix} QGL_4 & QGL_3 & QGL_2 \\ QGL_3 & QGL_2 & QGL_1 \\ QGL_2 & QGL_1 & QGL_0 \end{bmatrix}$$

Inductive Hypothesis: Assume for some $k \geq 1$:

$$S^k M = \begin{bmatrix} QGL_{k+3} & QGL_{k+2} & QGL_{k+1} \\ QGL_{k+2} & QGL_{k+1} & QGL_k \\ QGL_{k+1} & QGL_k & QGL_{k-1} \end{bmatrix}$$

Inductive Step ($n = k + 1$): Consider $S^{k+1}M = S \cdot (S^k M)$:

$$\begin{bmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} QGL_{k+3} & QGL_{k+2} & QGL_{k+1} \\ QGL_{k+2} & QGL_{k+1} & QGL_k \\ QGL_{k+1} & QGL_k & QGL_{k-1} \end{bmatrix}$$

Calculate each element systematically:

1. **First Row:**

$$(1,1):\ 2QGL_{k+3} - QGL_{k+1} = QGL_{k+4} \qquad \text{(by recurrence relation)}$$

$$(1,2):\ 2QGL_{k+2} - QGL_k = QGL_{k+3}$$

$$(1,3):\ 2QGL_{k+1} - QGL_{k-1} = QGL_{k+2}$$

2.Second Row:

$$(2,1):\ QGL_{k+3}$$

$$(2,2):\ QGL_{k+2}$$

$$(2,3):\ QGL_{k+1}$$

3. **Third Row:**

$$(3,1):\ QGL_{k+2}$$

$$(3,2):\ QGL_{k+1}$$

$$(3,3):\ QGL_k$$

Thus, the product matrix becomes:

$$\begin{bmatrix} QGL_{k+4} & QGL_{k+3} & QGL_{k+2} \\ QGL_{k+3} & QGL_{k+2} & QGL_{k+1} \\ QGL_{k+2} & QGL_{k+1} & QGL_{k} \end{bmatrix}$$

which exactly matches the required form for $n = k + 1$.

By mathematical induction, the theorem holds for all $n \geq 0$.

Theorem 9 Let QGL_n be the nth $G{-}L$ 3-parameter generalized quaternions. Then, the matrix representation of QGL_n can be written as follows:

$$M_{QGL_n} = \begin{bmatrix} GL_n & -\lambda_1\lambda_2 GL_{n+1} & -\lambda_1\lambda_3 GL_{n+2} & -\lambda_2\lambda_3 GL_{n+3} \\ GL_{n+1} & GL_n & -\lambda_3 GL_{n+3} & \lambda_3 GL_{n+2} \\ GL_{n+2} & \lambda_2 GL_{n+3} & GL_n & -\lambda_2 GL_{n+1} \\ GL_{n+3} & -\lambda_1 GL_{n+2} & \lambda_1 GL_{n+1} & GL_n \end{bmatrix},$$

where the matrix M_{QGLn} is called the fundamental matrix for G-L 3-parameter generalized quaternions.

Proof. By multiplying $QGL_n = GL_n + GL_{n+1}i + GL_{n+2}j + GL_{n+3}k$ with $1,i,j,k$ from the left side and using Table 1, we obtain:

$$QGLn1 = GLn + GLn{+}1i + GLn{+}2j + GLn{+}3k\ ,$$

$$QGLni = -\lambda1\lambda2GLn{+}1 + GLni + \lambda2GLn{+}3j - \lambda1GLn{+}2k\ ,$$

$$QGLnj = -\lambda1\lambda3GLn{+}2 - \lambda3GLn{+}3i + GLnj + \lambda1GLn{+}1k\ ,$$

$$QGLnk = -\lambda2\lambda3GLn{+}3 + \lambda3GLn{+}2i - \lambda2GLn{+}1j + GLnk\ .$$

Then, writing the coefficients of $\{1, i, j, k\}$ the above equations as a column gives the matrix in Theorem 9.

Corollary 10 Let QGL_n and QGL_m are two G-L 3-parameter generalized quaternions. Then, the following can be given:

$$M_{QGL_n} \cdot QGL_m^* = M_{QGL_n} \cdot QGL_m^* = (QGL_n \cdot QGL_m)^*,$$

where the superscript * represents column matrix forms.
Hence,

$$\mathrm{QGL}_m^* = [\mathrm{GL}_n \quad \mathrm{GL}_{n+1} \quad \mathrm{GL}_{n+2} \quad \mathrm{GL}_{n+3}]^T,$$

$$\mathrm{QGL}_m^* = [\mathrm{GL}_m \quad \mathrm{GL}_{m+1} \quad \mathrm{GL}_{m+2} \quad \mathrm{GL}_{m+3}]^T$$

and

$$(QGL_n{\cdot}QGL_m)^* = \begin{bmatrix} GL_nGL_{m+1} - \lambda_1\lambda_2 GL_{n+1}GL_{m+1} - \lambda_1\lambda_3 GL_{n+2}GL_{m+2} - \lambda_2\lambda_3 GL_{n+3}GL_{m+3} \\ GL_nGL_{m+1} + GL_{n+1}GL_{m+2} + \lambda_3 GL_{n+2}GL_{m+3} - \lambda_3 GL_{n+3}GL_{m+2} \\ GL_nGL_{m+2} + GL_{n+2}GL_{m+3} - \lambda_2 GL_{n+1}GL_{m+3} + \lambda_2 GL_{n+3}GL_{m+1} \\ GL_nGL_{m+3} + GL_nGL_{m+3} + \lambda_1 GL_{n+1}GL_{m+2} - \lambda_1 GL_{n+2}GL_{m+1} \end{bmatrix}.$$

Conclusion

With the continuous expansion of mathematical research, the integration of various sequences with algebraic structures has become an important pathway to new knowledge. This paper combines Gaussian Leonardo numbers with quaternions to construct a class of G-L 3-parameter generalized quaternions with three free parameters. This not only enriches quaternion theory but also opens new dimensions for sequence research.

During our investigation, we thoroughly explored the algebraic properties of G-L 3-parameter generalized quaternions, revealing key features such as recurrence relations, Binet-like formulas, generating functions, important identities, and summation formulas. These properties provide a foundation for theoretical studies of quaternions and potential applications. Additionally, using matrix representations of G-L 3-parameter generalized quaternions, we discovered a new identity. These results showcase the unique charm of G-L 3-parameter generalized quaternions and highlight the profound connections between mathematical structures.

In future research, readers may further explore applications of G-L 3parameter generalized quaternions in various disciplines, such as high-dimensional space descriptions in physics, complex data structure processing in computer science, and signal processing and system modeling in engineering. Investigating their interactions with other mathematical structures to address more complex practical problems is also worthwhile. We hope that subsequent work will deepen the understanding of G-L 3-parameter generalized quaternions, uncover more hidden properties, and apply them to broader fields, thereby promoting collaborative development in mathematics and related disciplines.

Future Research Directions

This work opens several promising avenues:

- **Quantum Applications:** Explore QGL_n in quantum gate representations and entanglement protocols using their SO(4) decomposition.
- **Generalized Sequences:** Investigate hybridizations with other sequences (e.g., Pell-Lucas, Jacobsthal) and higher-dimensional extensions.
- **Signal Processing:** Implement QGL-wavelet kernels in image compression and singularity detection, leveraging phase-shift properties.
- **Topological Insights:** Study knot invariants or topological phases using the Clifford algebra $C\ell(Q)$ framework.

Author contributions

Yong Deng and Jiangang Tang: Conceptualization, Methodology, Validation, Writing-original draft, Writing-review and editing. All authors contributed equally to this work.

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Declarations

Conflict of Interest No potential conflict of interest was reported by the authors. All authors of this manuscript have read and approved the final version submitted and contents of this manuscript have not been copyrighted or published previously and is notunder consideration for publication elsewhere.

Data Availability This manuscript presents purely theoretical mathematical research. No datasets were generated or analyzed during the current study. All results are derived from logical deduction and mathematical proof within the established axiomatic framework of ∞-category theory and condensed mathematics.

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