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Original Article

Diamond Hierarchical Network with Discrete, Unrelated, Self-Similar and Fractal Feature

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Abstract:
In this paper, a deterministic network called diamond hierarchical network (DHN) is first introduced. The topological properties of the network such as degree distribution, clustering coefficient, average path length and degree correlation are calculated analytically. The scaling of the average path length is the exponential growth of the total number of network nodes, the network does not have the nature of small-world. A small-world diamond hierarchical network (SWDHN) is defined according to the diamond hierarchical network, some basic properties of the network are studied, and the small-world phenomenon of SWDHN arises. Box dimension and fractality of DHN and SWDHN are considered.
Keywords: diamond hierarchical network; small-world diamond hierarchical network; topological characteristics; dimension

1. Introduction

Research on complex networks has been carried out in various fields such as economics, biology, physics, computer science, communications, sociology and so on. The study of the characteristics of network nature, such as complexity and dynamics, is an issue in all scientific fields. Hierarchical lattices have attracted much attention in the study of the statistical mechanics of phase transitions, because classical spin models, such as the Ising and Potts models, are exactly solvable for these lattices [1,2].

Topological properties of complex networks, such as scale-free degree distribution, small-world effect, fractal scaling and degree correlation, form the basis of our understanding of network organization [3-5]. Small-world effect and scale-free behaviour are two fundamental concepts. Small-world effect means that the expected number of edges required to connect any arbitrarily chosen node to another is small and grows at most logarithmically with the number of nodes. Scale-free behaviour means that the majority of nodes in the network have few connections to other nodes, while some nodes are joined to many other nodes in the network, which makes the network have a shorter average path length, resulting in small world effect [6]. At the same time, small world effect can also promote the formation of scale-free networks. The short path connection between nodes can increase the number of local neighbors of nodes, thus enhancing the degree of nodes. The growth of degree can conform to the power law distribution, and then form a scale-free network. According to the characteristics of social networks, Farhan and Gyu constructed a scale-free artificial social network model with small-world properties through the random walk and triangle generation scheme [7].

Network science has been a powerful tool for analyzing complex system in real life. The physical significance about the indicators of complex networks in seismicity was presented, and the physical properties of complex network parameters were further investigated to apply the complex network theory to geophysics [8]. Research had put forward the epidemic model of complex networks under constant control, and proved that the topology of complex networks had an important influence on the spread of epidemic under pulse control [9]. The properties of clustering coefficient, degree distribution and small-world feature were employed to investigate the spatial connections and architecture of precipitation networks in the Yellow River Basin, which could help researchers to study the spatial rainfall connections of the YRB [10].

In fact, the small-world nature and scale-free behaviour cannot provide sufficient characterization of the real system. It has been observed that complex systems in real life exhibit degree correlations among their nodes [11]. Degree correlation also plays an important role in the characterization of network topology. The topological properties of networks can be applied to the research of many real networks. The in vitro neuronal culture networks (microscale) and neuronal culture cluster networks (mesoscale) were reconstructed, the topological characteristics of the networks were explored, which demonstrated that mouse neurons showed self-optimised behaviour over time [12]. The topological characteristics of a Protein-Protein Interaction network had also been analyzed, and the results showed that this PPI network had the typical scale-free and small-world feature [13]. The force chain network topology graph (FCNTG) of aggregate blend at each compaction degree was constructed, the characteristics of each FCNTG were quantified using the indicators characterizing complex network, and the internal structural features of aggregate blend were analyzed using these topological feature indicators to provide a theoretical basis for their application in construction [14].

Recently, in the application of the renormalization process in different real networks, self-similarity emerges when the network is renormalize. When the boxes are transformed into new nodes of a smaller network during the coarsening process, self-similarity refers to the invariant scale-free distribution probability of finding a node of degree k, $P(k) \sim k^{-\gamma}$, i.e. the exponent γ remains constant when it has different box sizes under renormalization [15]. Self-similarity refers to the property of certain features in complex networks exhibiting similarities at different scales. Fractality also manifests as the similarity of certain parts at different length scales, fractal network models are self-similar, while self-similar networks are not always fractal [16]. The self-similarity of complex networks was measured by the classical distance of the nodes, and a box-covering algorithm with degree-degree distance was proposed to compute the value of dimension of the complex network as a way to explore the self-similarity of complex networks [17]. It is shown that self-similar scale-free networks are not assortative [18,19], and the characteristic of disassortativity is scale-invariant under renormalization. Soon-Hyung Yook et al. studied two genetic regulatory networks, showed their self-similarity and scale-free characteristics, and found that the qualitative feature of disassortativity was scale-invariant under renormalization [20].

The self-similarity and disassortativity of scale-free networks make such networks more robust against a sinister attack on large degree nodes, compared to the very vulnerable non-fractal scale-free network. The properties of fractal networks have attracted much attention. The properties of various real complex networks had been analyzed under length-scale transformations and the self-similarity and fractality of complex network had been explored [21]. Agata Fronczak et al. discussed that fractal networks possess local self-similarity and global scale invariance through microscopic and macroscopic indices, and verified these properties in real networks and several fractal network models [22]. Skums & Bunimovich proposed the theory of fractal graphs, which could provide effective information for inferring the formation mechanism of real-life networks and analyzed the genetic networks representing the structure of 323 intra-host Hepatitis C populations sampled at different stages of infection [23]. Many self-similar fractal networks were constructed, and a fractal network generated by the Sierpinski-like carpet had been analyzed. It was found that this network was scale-free and has the small-world effect, but it was not fractally scaled [24]. A evolving fractal networks was generated by Durer Pentagon. By discussing the topological properties of the network and utilizing self-similarity and the renewal theorem, an asymptotic formula for the average path length of the network was obtained

[25]. A fractal network was generated from a non-symmetric and self-similar planar fractal with Hausdoff dimension $\log_3 6$, the degree distribution, the average clustering coefficients and other properties of the network were analyzed with an in-depth exploration of the small-world effect and scale-free effect of fractal networks [26].

This paper introduces a family of deterministic networks known as diamond hierarchical network (DHN). DHN exhibits a discrete degree distribution, self-similarity, and degree-degree uncorrelation, but does not possess the small-world property. Additionally, we present the deterministic construction of a small-world diamond hierarchical network (SWDHN), which possess the basic structural properties including discreteness, self-similarity, and uncorrelation, leading to the small-world property.

The paper is structured as follows: Section 2 proposes the construction of the diamond hierarchical network. Section 3 examines its topological properties, including degree distribution, clustering coefficient, average path length (Apl), and degree correlation. Section 4 builds the deterministic construction of the SWDHN among the diamond hierarchical network and analyzes its topological properties. Section 5 computes the box dimensions of DHN and SWDHN, noting their self-similarity. Section 6 presents our conclusions.

2. Construction of Diamond Hierarchical Network

In many real networks, such as the World Wide Web, the number of nodes grows exponentially with time. For example, since 1990, the number of nodes on the WWW has increased from hundreds to thousands. The diamond hierarchical network, shown in Figure 1, is constructed iteratively. We denote the diamond hierarchical network by H(t) ($t \geq 0$) after t generations. The construction of the DHN is as follow: For t = 0, H(0) is an equilateral triangle with three vertices. For $t \geq 1$, H(t) is generated by placing three copies of H(t-1) on the outermost edges of H(t-1), with one copy located in the center. For $t \geq 1$, H(t) is generated by placing three copies of H(t-1) on the outermost edges of H(t-1), with one copy located in the center. This means that H(t) is formed from four copies of H(t-1), as shown in Figure 1.

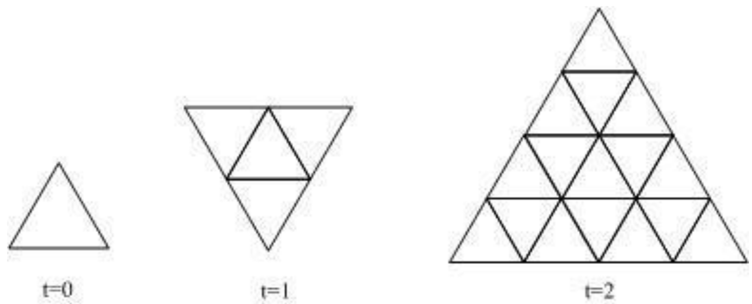

Figure 1: The construction of the DHN involves two initial steps.

The growing process is repeated t times to obtain the infinite DHN in the limit. Griffiths and Kaufman proposed two explanations for the construction of the hierarchical networks, namely aggregation and miniaturization[1]. These interpretations reflect the self-similar feature of the hierarchical network, allowing for the computation of their topological characteristics.

Topological characteristics of the network

The following steps involve computing the number of nodes and edges in network H(t). At step t, let $n_v(t)$ represent the total number of nodes and $n_e(t)$ represent the total number of edges. As per the growing process of the network, H(t) is iterated by H(t-1). Therefore, it is easy to identify the following relationships: $n_v(t+1) = 4n_v(t) - 3(2^t + 1)$ and $n_e(t+1) = 4n_e(t) - 3\cdot 2^t$. Based on Given the initial conditions of $n_v(0) = 3$ and $n_e(0) = 3$, it can be concluded that.

$$n_v(t) = 2^{2t-1} + 3\cdot 2^{t-1} + 1 \tag{1}$$

and $$n_e(t) = 3\cdot 2^{2t-1} + 3\cdot 2^{t-1},\ t \geq 1 \quad (2)$$

For large t, the average degree $\langle k\rangle = \frac{2n_e(t)}{n_v(t)}$ is approximately 6. This indicates that the network is as sparse as most real systems.

Degree distribution

Degree distribution is an important topological property. Let the degree of a node at step t be denoted as $k_i(t)$. By construction, the following relations can be easily obtained:

$$k_i(t) = \begin{cases} k_i(t-1)+2, & k_i(t-1)=2 \\ 6, & k_i(t-1)=4,6 \end{cases} \quad (3)$$

When a new node *i* is added to the network at step $t_i(t_i \geq 1)$, its degree is initially degree of 2. At step *t*, the degree of node *i* increases to:

$$k_i(t) = 2(t - t_i + 1),\ |t - t_i| \leq 2 \quad (4)$$

The degree of any node in the DHN is limited to $k \leq 6$. At step *t*, $n_{v,2}(t), n_{v,4}(t)$ and $n_{v,6}(t)$ represent the number of nodes with degree 2, 4, and 6, where $n_{v,2}(t) = 3, n_{v,4}(t) = 3(2^t - 1)$ and $n_{v,6}(t) = 2^{2t-1} - 3\cdot 2^{t-1} + 1, t \geq 1$, respectively. Therefore, the degree distribution can be expressed as follows:

$$P(k=2) = \frac{n_{v,2}(t)}{n_v(t)} = \frac{3}{2^{2t-1} + 3\cdot 2^{t-1} + 1}$$

$$P(k=4) = \frac{n_{v,4}(t)}{n_v(t)} = \frac{3(2^t - 1)}{2^{2t-1} + 3\cdot 2^{t-1} + 1}$$

and $$P(k=6) = \frac{n_{v,6}(t)}{n_v(t)} = \frac{2^{2t-1} - 3\cdot 2^{t-1} + 1}{2^{2t-1} + 3\cdot 2^{t-1} + 1}$$

As t approaches infinity, the probabilities of P(k = 2) and P(k = 4) tend towards zero, while P(k = 6) approaches a non-zero value of 1. The degree distribution is observed to follow a discrete distribution.

Clustering coefficient

The clustering coefficient is a significant characteristic of a network as it measures the level of cohesiveness around a particular node. The clustering coefficient of a node *i* with degree k_i is defined as $C_i = \frac{2e_i}{k_i(k_i - 1)}$, where e_i is the number of edges between the nodes in the neighborhood of k_i. The clustering coefficient of the entire network is calculated as the average of all individual $C_i's$. The vertices of the external triangle in *H(t)* have a degree of 2, resulting in a clustering coefficient of 1 for these three nodes. The remaining nodes have degrees of 4 and 6, with clustering coefficients of 0.5 and 0.4, respectively. The clustering coefficient for a single node with degree *k* is denoted as *C(k)*. After t generations of evolution, the clustering coefficient *Ct* can be calculated as follows:

$$C_t = \frac{\sum_i C(i)}{n_v(t)} = \frac{3\cdot 1 + 3\cdot(2^{t-1} - 1)\cdot 0.5 + (2^{2t-1} - 3\cdot 2^{t-1} + 1)\cdot 0.4}{2^{2t-1} + 3\cdot 2^{t-1} + 1}.$$

For large values of *t*, the equation above converges to a non-zero value of $C = 0.4$. This indicates that the network is highly clustered.

Average Path Length

The average path length is a crucial property for both communication and transportation networks. We represent the shortest path lengths of *H(t)* as a matrix, where the entry d_{ij} represents the geodesic path from node *i* to node *j*. The geodesic path is the path with the minimum length connecting two nodes. The average path length $\overline{d}_t$, also known as the characteristic path length, is used to measure the typical separation between two nodes in *H(t)*. It is defined as the average geodesic length over all pairs of nodes. This metric provides a clear and objective evaluation of the network's structure.

$$\overline{d}_t = \frac{2D_t}{n_v(t)(n_v(t)-1)} \tag{5}$$

where *Dt* represents the sum of the total distances between two nodes over all pairs, i.e. $D_t = \sum_{i,j \in H(t)} d_{ij}$. An exact solution for Apl has been obtained.

$$\overline{d}_t = \frac{\frac{33}{10240}\cdot 2^{5t} + \frac{35}{768}\cdot 2^{4t} + \frac{37}{128}\cdot 2^{3t} + \frac{7201}{768}\cdot 2^{2t} + \frac{1}{16}\cdot 2^{2t}\cdot t + \frac{11}{20}\cdot 2^{t}}{(2^{2t-1}+3\cdot 2^{t-1}+1)(2^{2t-1}+3\cdot 2^{t-1})} \tag{6}$$

Equation (6) shows that Apl scales as $\frac{33}{2560}\cdot 2^t$ in the infinite limit of network order. The exponential growth of network size $n_v(t)$ is evident from *dt*. Since $n_v(t) \sim 4^t$ becomes large for large *t*, we have $\overline{d}_t \sim n_v(t)^{\frac{1}{2}}$, which means that the DHN cannot be considered a small world.

Degree Correlation

The field of complex networks has made significant progress, with particular interest in degree correlation due to its ability to give rise to interesting network structure effects. The average degree of the nearest neighbors for nodes with degree *k*, denoted as $k_{nn}(k)$, is a useful quantity for characterizing degree-degree correlation. This is defined by (Pastor-Satorras et al., 2001)

$$k_{nn}(k) = \sum_{k'} k' P(k' \mid k). \tag{7}$$

Degree correlation is a function of the node's degree k. An increase in $k_{nn}(k)$ with *k* indicates that nodes tend to connect with others of similar or larger degree. If this is the case, the network is considered assortative. Conversely, a decrease in $k_{nn}(k)$ with *k* characterises a disassortative network, where nodes with large degrees are more likely to have near neighbours with small degrees. If there is no correlation, $k_{nn}(k) = const$.

The value of $k_{nn}(k)$ for the DHN can be calculated precisely. This is due to their construction, which ensures that their neighbors have a degree of 4. It follows that $k_{nn}(2) = 4$.

It has been found that the degree distribution of the network is discrete and there is no degree correlation. The value of $k_{nn}(k)$ is independent of *k*, as shown in equation (7).

$$k_{nn}(k) = \langle k^2 \rangle / \langle k \rangle \tag{8}$$

To obtain information on how the degree is distributed among these nodes in an undirected network, you can either use the degree distribution *P(k)* or calculate the moments of the distribution. The n-moments of *P(k)* are defined as

$$\langle k^n \rangle = \sum_k k^n P(k). \quad (9)$$

So the second moment is

$$\langle k^2 \rangle = \sum_k k^2 P(k) = 2^2 P(2) + 4^2 P(4) + 6^2 P(6)$$
$$= \frac{36 \cdot 2^{2t-1} - 12 \cdot 2^{t-1}}{2^{2t-1} + 3 \cdot 2^{t-1} + 1}$$

By substituting the above equation into equation (8), we can derive

$$k_{nn}(k) = \frac{6 \cdot 2^{2t-1} - 2^t}{2^{2t-1} + 2^{t-1}} \quad (10)$$

For large *t*, degree correlation is constant, indicating that $k_{nn}(k)$ is uncorrelated with nodes of degree *k* in the DHN. Degree correlation can be described by a Pearson correlation coefficient *r* of the degrees at either end of a link.

$$r = \frac{\langle k \rangle \langle k^2 k_{nn}(k) \rangle - \langle k^2 \rangle^2}{\langle k \rangle \langle k^3 \rangle - \langle k^2 \rangle^2} \quad (11)$$

If the network is disassortative, the correlation coefficient is $r < 0$, whereas assortative graphs have a value of $r > 0$. Equation (11) shows that the correlation coefficient is equal to zero, indicating that the network is uncorrelated.

Small-world Diamond Hierarchical Network

This subsection focuses on the construction and properties of the SWDHN and its small-world property. The goal is to reduce the diameter while maintaining the original graph structure. To achieve this, a new central node is added to the previous DHN and connected to a specific set of original nodes, as described in reference[27].

The SWDHN is presented as *SWHm(t)*, where $t \geq 2$ and $m = 1, 2, \cdots, t-1$, *H(t)* can be viewed as 4^{t-m} copies of *H(m)*, with certain nodes identified. *SWHm(t)* is the graph obtained from *H(t)*, with the central node connected to the vertices of the external triangle of all copies of *H(m)* (refer to Figure 2).

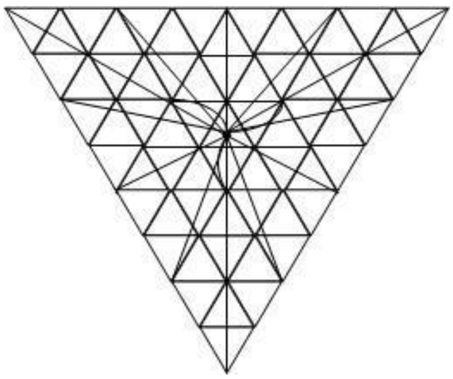

Figure 2: The network described is a small-world hierarchical network with t = 4 and m = 1.

The order of *SWHm(t)* is equal to one plus the order of *H(t)*. The size of *SWHm(t)* is equal to the size of H(t) plus the number of added edges. It is evident that the number of added edges is the order of *H(t − m)*, which represents the degree of the central node.

$$k(t) = n_v(t-m) = 2^{2t-2m-1} + 3 \cdot 2^{t-m-1} + 1$$

Based on equations (1) and (2), it can be determined that *SWHm(t)* has a specific order and size.

$$N_v(t) = n_v(t) + 1 = 2^{2t-1} + 3 \cdot 2^{t-1} + 2 \quad (12)$$

and

$$N_e(t) = n_e(t) + k(t) = 2^{2t-1}(3 + 2^{-2m}) + 3 \cdot 2^{t-1}(1 + 2^{-m}) + 1. \quad (13)$$

So for large *t*, the average degree of *SWHm(t)* is approximately $6 + 2^{1-3m}$, which is a function of m. The average degree gradually decreases with an increase in m.

The central node has a clustering coefficient of 0. The vertices of the external triangle have a degree of 3 and a clustering coefficient of $\frac{1}{3}$. There are $3(2^{t-m} - 1)$ nodes with a degree of 5 and a clustering coefficient of $\frac{3}{10}$. Let $N_{v,4}(t)$ be the number of nodes with a degree of 4 and a clustering coefficient of $\frac{1}{2}$, and $N_{v,6}(t)$ be the number of nodes with a degree of 6 and a clustering coefficient of $\frac{2}{5}$. If all remaining nodes have a degree of 7, then the number of nodes is $N_{v,7}(t)$ and the clustering is $\frac{2}{7}$. According to the construction of *SWHm(t)*, this is the case.

$N_{v,4}(t) = n_{v,4}(t) - 3(2^{t-m} - 1) = 3 \cdot 2^t - 3 \cdot 2^{t-m}$,

$N_{v,7}(t) = k(t) - 3 - 3(2^{t-m} - 1) = 2^{2t-2m-1} - 3 \cdot 2^{t-m-1} + 1$

and

$N_{v,6}(t) = n_{v,6}(t) - N_{v,7}(t) = 2^{2t-1}(1 - 2^{-2m}) - 3 \cdot 2^{t-1}(1 - 2^{-m})$

This provides the value for clustering

$$C_t = \frac{3 \cdot \frac{1}{3} + N_{v,4}(t) \cdot \frac{1}{2} + N_{v,5}(t) \cdot \frac{3}{10} + N_{v,6}(t) \cdot \frac{2}{5} + N_{v,7}(t) \cdot \frac{2}{7}}{2^{2t-1} + 3 \cdot 2^{t-1} + 2}. \quad (14)$$

For large values of *Nv(t)*, Ct approaches a non-zero value of $\frac{2}{5} - \frac{4}{35} \cdot 2^{-2m}$. For any $1 \le m \le t-1$, the clustering of *SWHm(t)* is slightly smaller than the clustering of *H(t)*.

The diameters of *SWHm(t)* and *H(t)* are denoted by *Diam(SWHm(t))* and *Diam(H(t))*, respectively. To calculate *Diam(SWHm(t))*, we can observe that in *H(m)*, the maximum distance is at most *Diam(H(m))/2* from the set of vertices of the external triangles. It is important to note that *Diam(H(t))* is always equal to 2t. Therefore, an upper bound for *Diam(SWHm(t))* is *Diam(H(m)) + 2*. This is also a lower bound, so we can conclude that *Diam(SWHm(t))* is equal to *2m + 2*. It is worth noting that the diameter of *SWHm(t)* is solely dependent on the value of *m*. If m is less than or equal to $\log_2^t$, the order of *SWHm(t)* is $N_v(t) = 2^{2t-1} + 3 \cdot 2^{t-1} + 2$. In this case, the diameter of *SWHm(t)* is at most *t+2*. The diameter of *SWHm(t)* increases logarithmically with the network order, while the clustering coefficient remains relatively constant. Therefore, *SWHm(t)* exhibits the small-world property, which distinguishes it from *H(t)* [3].

The degree distribution of SWHm(t) is discrete due to the central node's minimal impact on the degree of other nodes. And degree correlation $k_{nn}(k)$ of $SWH_m(t)$ is independent of *k*, so the network is uncorrelated.

Dimension and Fractality

According to the box-counting method, we can find the box-counting dimension. The box dimension d_B is given by

$$N_B(l_B) \approx l_B^{-d_B} \quad (15)$$

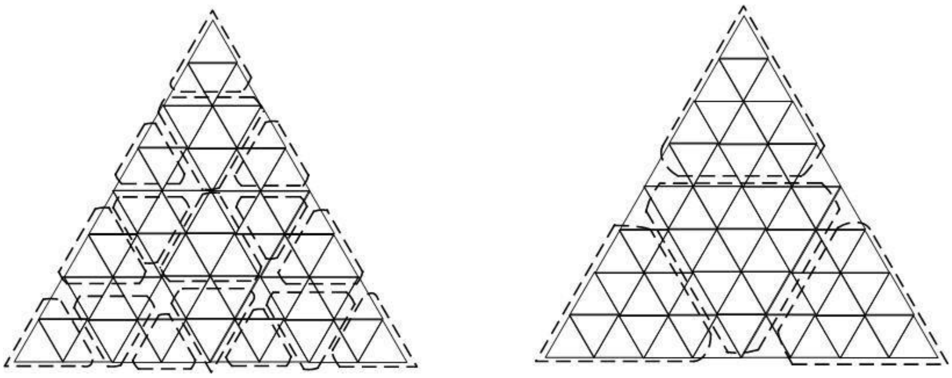

Figure 3: For t = 3, the number of boxes required to cover *H(t)* with boxes of linear size l_B is $N_B(l_B)$ = 16(l_B = 3) and $N_B(l_B)$ = 4(l_B = 5), respectively.

The number of boxes required to cover the graph can be determined by counting the boxes of linear size l_B used to cover *H(t)* (refer to figure 3). Here, the linear size of a box is defined as one plus its diameter. The table 1 shows the number of boxes N_B required for different values of l_B.

Table1: the values of l_B and $N_B(l_B)$

l_B	2	3	5	…	2^{t+1}
$N_B(l_B)$	4^t	4^{t-1}	4^{t-2}	…	1

The box-counting dimension value is $d_B = \frac{ln4}{ln2} = 2$, the network presents a fractal topology [28].

The process of determining the box-counting dimension of *SWHm(t)* is similar to that of computing the box-counting dimension of *H(t)*. Both can be covered with boxes of linear size l_B. To obtain the same number of boxes for every size l_B, we only need to add the central node to one of the boxes. Therefore, the box-counting dimensions of *SWHm(t)* and *H(t)* are identical.

The fractal dimension of a hierarchical network can be found by following an inverse renormalization procedure. Mathematical framework can be used to derive the necessary equations for large *t*:

$$Diam[H(t)] = 2Diam[H(t-1)] \text{ and } n_v(t) \approx 4n_v(t-1).$$

The equations above describe the change in diameter and order of the DHN *H(t)* during the generation process. From the above equations, it can be observed that the quantities *Diam(H(t))* and $n_v(t)$ increase by a factor of 2 and 4, respectively. Therefore, for any times t1 and t2 (where $t_1 < t_2$), the following relations can be obtained:

$$Diam[H(t_2)] = 2^{t_2-t_1} Diam[H(t_1)] \quad (16)$$

and

$$n_v(t_2) \approx 4^{t_2-t_1} n_v(t_1) \quad (17)$$

The scaling exponent can be derived in terms of the microscopic parameters from equations (16) and (17) dimension is $d_B = \frac{ln4}{ln2} = 2$ [29], and the same value is obtained in the calculation of the box-counting method.

Conclusions

The diamond hierarchical network, constructed deterministically, possesses several intriguing topological properties. Its degree distribution is discrete, and its clustering coefficient is 0.4. However, its average path length exhibits exponential growth with network size, indicating that it is not a small-world network. The network is uncorrelated based on degree correlation and Pearson coefficient *r*. It has a fractal topology with a general fractal dimension of 2. These results suggest that the uncorrelated feature can be easily understood based on the growth process of DHN. As the network size increases, many nodes do not change in degree.

The study examines the SWDHN based on the DHN. It discusses the degree distribution, clustering coefficient, diameter, and fractal dimension of SWDHN, and observes the small-world phenomenon. The text also notes that self-similarity and the small-world property may have a significant impact on other dynamic processes. In the future, it is important to study all types of hierarchical networks. It would be particularly interesting to apply complex network theory in practical applications.

Disclosure Statement

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Data Availability Statement

Data sharing is not applicable to this article to this article as no new data were created or analyzed in this study.

Contributions

Min Wang devised the project, the main conceptual ideas, proof outline, and wrote the manuscript. Daping Tian verified the experimental results, reviewed and edited the article. The authors have reviewed the final version of the article, agreed to its publication and take responsibility and accountability for its contents.

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