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ORIGINAL ARTICLE

Exponential Powers of the Product of Two Univalent Log-Harmonic Mappings

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Abstract
In this paper, we examine some classes of univalent λ-spirallike log-harmonic mappings defined on the unit disk. We present several sufficient conditions for two given λ-spirallike log-harmonic mappings, f_1 and f_2, which guarantee that exponential powers of their product, $F(z)=\left(f_1(z)f_2(z)\right)^{e^{-i\lambda}}$, is a univalent starlike log-harmonic mapping.

Keywords: log-harmonic mapping, starlike mapping, λ-spirallike log-harmonic mappings.

Introduction

1. Introduction and preliminary results

Let U denote the open unit disk $\mathrm{U}=\{z\in\mathbb{C}:|z|<1\}$, and let $\mathrm{H}(\mathrm{U})$ represent the linear space of all holomorphic functions defined on U. In recent years, the study of harmonic and log-harmonic mappings has attracted considerable attention due to their rich geometric properties and applications in complex analysis and geometric function theory.A log-harmonic mapping f defined on U is a solution of the nonlinear elliptic partial differential equation

- $$\overline{f_{\bar{z}}(z)}=\mu(z)\left(\frac{\overline{f(z)}}{f(z)}\right)f_z(z) \qquad (0.1)$$

where μ denotes the second complex dilatation of f, and $\mu\in\mathrm{H}(\mathrm{U})$ such that $|\mu(z)|<1$ for all $z\in\mathrm{U}$. This class of mappings has been extensively studied due to their connections with quasiconformal mappings and spirallike functions (see [1–3]).

Recently, significant progress has been made in understanding the properties of log-harmonic mappings, particularly those that are univalent and sense-preserving. For instance, Abdul- hadi and Bshouty [1] investigated the univalence criteria for log-harmonic mappings in the unit disk, while Abdulhadi and Hengartner [2] explored the polynomial structure of such mappings. The study of spirallike log-harmonic mappings was initiated by Abdulhadi and Hengartner[3], where they established the basic properties and characterizations of these mappings. Further developments in this direction were made by AbdulHadi, Alareefi and Ali [7], who explored the convex-exponent product and other geometric properties of logharmonic mappings.

The Jacobian J_f of f is given by

- $$J_f=|f_z|^2-|f_{\bar{z}}|^2=|f_z|^2(1-|\mu|^2),$$

which is positive, and therefore, every non-constant log-harmonic mapping is sense-preserving.

Let B represent the set of bounded analytic functions $|\mu(z)|<1$ in U. Let B_0 denote

its subclass consisting of $\mu \in B$ with $\mu(0)=0$. Furthermore, as discussed in [1,2], if f is a non-vanishing log-harmonic mapping that vanishes only at $z=0$, then f should be in the form

- $$f(z)=z^m|z|^{2m\beta}h(z)\overline{g(z)} \qquad (0.2)$$

where m is a non-negative integer, $\mathrm{Re}\beta>-\frac{1}{2}$, and h and g are analytic functions in $H(U)$ satisfying $h(0) \neq 0$ and $g(0) = 1$ (see [1]). The exponent β in the above equation depends only on $\mu(0)$ and is given by

- $$\beta=\overline{\mu(0)}\frac{1+\mu(0)}{1-|\mu(0)|^2}.$$

It is noted that $f(0)$ is nonzero if and only if m = 0. Conversely, a univalent log-harmonic mapping in U vanishes at the origin only if m=1. That is, f has the form

- $$f(z)=z|z|^{2\beta}h(z)\overline{g(z)},$$

- Where $\mathrm{Re}\beta>-\frac{1}{2}$ and $0\notin(hg)(U)$. This

subclass has been extensively studied in recent years, for instance, in [4, 5, 8, 13, 14].

In this work, we focus on univalent and sense-preserving log-harmonic mappings in U with respect to $\mu\in B_0$. These mappings are of the form

$$f(z)=zh(z)\overline{g(z)},$$

If $f(z)=zh(z)\overline{g(z)}$ is univalent and satisfies the condition

- $$|\mu(z)|=\left|\frac{\overline{f_{\bar z}(z)}/\overline{f(z)}}{f_z(z)/f(z)}\right|=\left|\frac{zg'(z)/g(z)}{1+zh'(z)/h(z)}\right|\le k<1,$$

we call $f(z)$ a K-quasi-conformal log-harmonic mapping on U, where $K=\frac{1+k}{1-k}$.

The criteria for univalence and quasiconformal extension introduced in [11, 12] provide significant tools for understanding these mappings.

Definition 1. Let be a univalent log-harmonic mapping. We say that f is a starlike logharmonic mapping of order α if

- $$\frac{\partial\left(\arg f(re^{i\theta})\right)}{\partial\theta}=\mathrm{Re}\left(\frac{zf_z-\bar z f_{\bar z}}{f}\right)>\alpha, \quad (0\le\alpha<1)$$

for all $z\in U$ Denote by $S^*_{LH}(\alpha)$ the subclass of all starlike log-harmonic mappings of order α. When α = 0, we denote by S^*_{LH} the subclass of all starlike log-harmonic mappings.

By setting β = 0 and g(z) = 1 in Definition 1, we derive the subclass of starlike analytic functions of order α in A, which we denote as $S^*(\alpha)$.

The subsequent lemma elucidates the relationship between the classes $S^*_{LH}(\alpha)$ and $S^*(\alpha)$.

Lemma 1.1. (See[3, Lemma 2.4]) Let be $f(z)=zh(z)\overline{g(z)}$, log-harmonic mapping in U, with $0\notin hg(U)$ .Then $f\in S^*_{LH}(\alpha)$ if and only if $\varphi(z)=zh(z)/g(z)\in S^*(\alpha)$

In [3], the authors investigated the class of λ-spirallike functions and established the following: Let λ be a real number such that log-harmonic mapping on U with $0\notin hg(U)$. Then f is a λ-spirallike log-harmonic mapping if it satisfies the following condition:

- $$\mathrm{Re}\left(e^{-i\lambda}\frac{zf_z-\bar z f_{\bar z}}{f}\right)>0, \quad 0\le\alpha<1$$

for all $z\in U$. We note that a simply connected domain Ω in C containing the origin is said to be λ-spirallike, $|\lambda|<\frac{\pi}{2}$, if $\omega\exp(-te^{i\alpha})\in\Omega$ for all $t>0$ whenever $\omega\in\Omega$.

Furthermore, f is a λ -spirallike function if $f(\mathrm{U})$ is a λ -spirallike domain. Motivated by this, we define the class of λ -spirallike log-harmonic mappings of order α as follows:

Definition 2. Let $f(z)=z|z|^{2\beta}h(z)\overline{g(z)}$ is a log-harmonic mapping on U, with $0\notin hg(\mathrm{U})$. Then, we say that f is a λ-spirallike log-harmonic mapping of order $\alpha(0\le\alpha<1)$ if

- $$\mathrm{Re}\left(e^{-i\lambda}\frac{zf_z-\overline{z}f_{\overline{z}}}{f}\right)>\alpha\cos\lambda,\quad (z\in\mathrm{U})$$

for some real number λ with $|\lambda|<\frac{\pi}{2}$. The class of these functions is denote by $\mathrm{SP}_{LH}^{\alpha}(\lambda)$.Furthermore, we define $\mathrm{SP}_{LH}^{1}(\lambda)=\cap_{0\le\alpha<1}\mathrm{SP}_{LH}^{\alpha}(\lambda)$.

Additionally, we denote by $\mathrm{S}^{\alpha}(\lambda)$ the subclass of all $f\in\mathrm{A}$ such that f is λ -spirallike of order α, and we define $\mathrm{S}^{\alpha}(1)=\cap_{0\le\alpha<1}\mathrm{S}^{\alpha}(\lambda)$.

Harmonic and log-harmonic mappings have been extensively studied, with notable contributions made in the areas of quasiconformal extensions and spirallike properties (see [3, 10, 12]). In[3], the authors detailed the properties of log-harmonic mappings and their relationships with spirallike functions as follows:

Lemma 1.2. ([3, Lemma 2.4]) Let $f(z)=z|z|^{2\beta}h(z)\overline{g(z)}$ be a log-harmonic mapping on U,with $0\notin hg(\mathrm{U})$ and $\mathrm{Re}\beta>-0.5$, then $f\in\mathrm{SP}_{LH}^{\alpha}(\lambda)$ if and only if :

- $$\psi(z)=\frac{zh(z)}{g(z)^{e^{2i\lambda}}}\in\mathrm{S}^{\alpha}(\lambda)$$

Expanding on the characterization of log-harmonic mappings, Basgöze and Keogh [10] establish a connection between these mappings and starlike functions, as detailed in the subsequent lemma:

Lemma 1.3. ([10, Lemma 1]) $f(z)\in\mathrm{SP}(\lambda)$ if and only if there is a $g(z)\in\mathrm{S}^{*}$ such that

- $$\left(\frac{f(z)}{z}\right)^{e^{-i\lambda}}=\left(\frac{g(z)}{z}\right)^{\cos\lambda},\qquad(0.3)$$

where the branches are chosen so that each side of the equation has the value 1 when $z=0$.

The proof of our theorem hinges on the following lemma, which establishes a critical relationship for log-harmonic mappings under specific conditions:

Lemma 1.4. Let $f(z)=zh(z)\overline{g(z)}$ is a log-harmonic mapping on U, and with respect to $\mu(z)\in\mathrm{B}_0$. If

- $$zh(z)g(z)=\phi(z)\qquad(0.4)$$

Then

- $$1+\frac{zh'(z)}{h(z)}=\frac{1}{1+\mu(z)}\frac{z\phi'(z)}{\phi(z)}.\qquad(0.5)$$

Proof. Taking the logarithm derivative of both sides of equation (1.4) yields

$$\frac{z\phi'(z)}{\phi(z)}=1+\frac{zh'(z)}{h(z)}+\frac{zg'(z)}{g(z)}=\left(1+\frac{zh'(z)}{h(z)}\right)(1+\mu(z))$$

Rearranging this equation gives us

- $$1+\frac{zh'(z)}{h(z)}=\frac{1}{1+\mu(z)}\frac{z\phi'(z)}{\phi(z)}.$$

Therefore, equation (1.5) is derived, completing the proof.

To provide a clear framework for the various subclasses of log-harmonic mappings that we will discuss, we introduce the following notations:

Definition 3. For $k\in(0,1], 0\le\alpha<1$, $\phi(z)\in\mathrm{A}$, we consider the following subclasses:

$$\mathrm{SP}_{LH}(\lambda,k):=\left\{f(z)=zh(z)\overline{g(z)}\in\mathrm{SP}_{LH}(\lambda),|\mu(z)|\le k<1\right\},$$
$$\mathrm{S}_{LH}^{*}(k):=\left\{f(z)=zh(z)\overline{g(z)}\in\mathrm{S}_{LH}^{*},|\mu(z)|\le k<1\right\},$$

$$\mathrm{SP}_{LH}(\lambda,k,\alpha):=\left\{f(z)=zh(z)\overline{g(z)}\in \mathrm{SP}_{LH}^{\alpha}(\lambda),|\mu(z)|\le k<1\right\},$$
$$\mathrm{SP}_{LH}^{*}(k,\alpha):=\left\{f(z)=zh(z)\overline{g(z)}\in \mathrm{S}_{LH}^{*}(\alpha),|\mu(z)|\le k<1\right\},$$
$$\mathrm{SP}_{LH}(\lambda,k,\phi):=\left\{f(z)=zh(z)\overline{g(z)}\in \mathrm{SP}_{LH}(\lambda),|\mu(z)|\le k<1, zh(z)g(z)=\phi(z)\right\}$$

In this paper, we focus on the exponential powers of the product of two univalent log-harmonic mappings, specifically those that are λ-spirallike. Our primary goal is to establish sufficient conditions under which the exponential power of their product remains univalent and starlike. This work extends the existing literature on the products of harmonic and log-harmonic mappings (see [7]) and provides new insights into the geometric properties of these mappings.

The following sections are arranged as follows: Section 1 provides an introduction to some foundational results. In Section 2, we establish sufficient conditions for the univalence of the exponential powers of the products of certain log-harmonic mappings. Having reviewed preliminary definitions and results.

2. Main Results

We now delve into the specific conditions for K-quasi-conformal starlike log-harmonic mappings. To establish sufficient conditions for the univalence of K-quasi-conformal starlike log-harmonic mappings, we present the following theorem:

Theorem 2.1. Let $f_j(z)=zh_j(z)\overline{g_j(z)}\in \mathrm{SP}_{LH}(\lambda,k)$ for $j=1,2$, and with respect to the same $\mu(z)\in \mathrm{B}_0$. Let λ be a real number such that $|\lambda|<\dfrac{\pi}{2}$. Then $F(z)=\left(f_1(z)f_2(z)\right)^{e^{-i\lambda}}\in \mathrm{S}_{LH}^{*}(k)$.

Proof. First, we show that the second dilatation of F, i.e., $|\hat{\mu}(z)|\le k<1$. Since

- $$\frac{F_z}{F}=e^{-i\lambda}\left(\frac{(f_1)_z}{f_1}+\frac{(f_2)_z}{f_2}\right),\quad \frac{F_{\bar z}}{F}=e^{-i\lambda}\left(\frac{(f_1)_{\bar z}}{f_1}+\frac{(f_2)_{\bar z}}{f_2}\right) \qquad (2.1)$$

and

- $$\mu(z)=\frac{\overline{\left(\frac{(f_1)_{\bar z}}{f_1}\right)}}{\frac{(f_1)_z}{f_1}}=\frac{\overline{\left(\frac{(f_2)_{\bar z}}{f_2}\right)}}{\frac{(f_2)_z}{f_2}}. \qquad (2.2)$$

Combining (2.1) and (2.2), we can derive

- $$\hat{\mu}(z)=\frac{\overline{\left(\frac{F_{\bar z}}{F}\right)}}{\frac{F_z}{F}}=\frac{e^{i\lambda}\left(\overline{\left(\frac{(f_1)_{\bar z}}{f_1}\right)}+\overline{\left(\frac{(f_2)_{\bar z}}{f_2}\right)}\right)}{e^{-i\lambda}\left(\frac{(f_1)_z}{f_1}+\frac{(f_2)_z}{f_2}\right)}=e^{2i\lambda}\mu(z)\frac{\frac{(f_1)_z}{f_1}+\frac{(f_2)_z}{f_2}}{\frac{(f_1)_z}{f_1}+\frac{(f_2)_z}{f_2}}=e^{2i\lambda}\mu(z). \qquad (2.3)$$

Then we have

- $$|\hat{\mu}(z)|=\left|e^{2i\lambda}\right||\mu(z)|\le k.$$

This implies that F is a K-quasi-conformal log-harmonic mapping. Now, we show that

- $$F(z)=\left(f_1(z)f_2(z)\right)^{e^{-i\lambda}}\in \mathrm{S}_{LH}^{*}. \qquad (2.4)$$

Since $f_j(z)=zh_j(z)\overline{g_j(z)}\in SP_{LH}(\lambda,k),(j=1,2)$, straightforward calculations yield

$$\mathrm{Re}\left(\frac{zF_z-\overline{z}F_{\overline{z}}}{F}\right)=\mathrm{Re}\left(e^{-i\lambda}\frac{z(f_1)_z-\overline{z}(f_1)_{\overline{z}}}{f_1}\right)+\mathrm{Re}\left(e^{-i\lambda}\frac{z(f_2)_z-\overline{z}(f_2)_{\overline{z}}}{f_2}\right)>0.$$

Hence F is starlike, and we deduce that $F\in S^*_{LH}(k)$. The proof is completed.

Building on the results established in Theorem 2.1, we now extend our findings to a more general case involving multiple log-harmonic mappings. This extension leads us to the fol- lowing corollary:

Corollary 2.2. Let $f_j(z)=zh_j(z)\overline{g_j(z)}\in SP_{LH}(\lambda,k)$ for $j=1,2,\cdots,n$, with respect to the same $\mu(z)\in B_0$ .Le λ be a real number such that $|\lambda|<\frac{\pi}{2}$,Then $F(z)=\left(f_1(z)f_2(z)\cdots f_n(z)\right)^{e^{-i\lambda}}\in S^*_{LH}(k)$

Proof. After simple calculations, we find that

$$|\hat{\mu}(z)|=\left|\frac{\frac{\overline{F_{\overline{z}}}}{\overline{F}}}{\frac{F_z}{F}}\right|=\left|\frac{e^{i\lambda}\left(\frac{\overline{(f_1)_{\overline{z}}}}{\overline{f_1}}+\frac{\overline{(f_2)_{\overline{z}}}}{\overline{f_2}}+\cdots+\frac{\overline{(f_n)_{\overline{z}}}}{\overline{f_n}}\right)}{e^{-i\lambda}\left(\frac{(f_1)_z}{f_1}+\frac{(f_2)_z}{f_2}+\cdots+\frac{(f_n)_z}{f_n}\right)}\right|$$

$$=\left|e^{2i\lambda}\mu(z)\frac{\frac{(f_1)_z}{f_1}+\frac{(f_2)_z}{f_2}+\cdots+\frac{(f_n)_z}{f_n}}{\frac{(f_1)_z}{f_1}+\frac{(f_2)_z}{f_2}+\cdots+\frac{(f_n)_z}{f_n}}\right|$$

$$=\left|e^{2i\lambda}\mu(z)\right|<k$$

This implies that F is a K-quasi-conformal mapping. Since

$$\mathrm{Re}\left(\frac{zF_z-\overline{z}F_{\overline{z}}}{F}\right)=\mathrm{Re}\left(e^{-i\lambda}\frac{z(f_1)_z-\overline{z}(f_1)_{\overline{z}}}{f_1}\right)+\mathrm{Re}\left(e^{-i\lambda}\frac{z(f_2)_z-\overline{z}(f_2)_{\overline{z}}}{f_2}\right)$$

$$+\cdots+\mathrm{Re}\left(e^{-i\lambda}\frac{z(f_n)_z-\overline{z}(f_n)_{\overline{z}}}{f_n}\right)>0$$

F is starlike, and we deduce that $F\in S^*_{LH}(k)$. The proof is completed.

Extending the results of Theorem 2.1, we now consider the scenario where the mappings involve distinct dilatations:

Theorem 2.3. Let $f_j(z)=zh_j(z)\overline{g_j(z)}\in SP_{LH}(\lambda,k)$ with respect to the $\mu_j(z)\in B_0(j=1,2)$, and let λ be a real number such that $|\lambda|<\frac{\pi}{2}$. If

$$\mathrm{Re}\left[\left(k^2-\mu_1\overline{\mu_2}\right)\left(1+\frac{zh_1'}{h_1}\right)\left(\overline{1+\frac{zh_2'}{h_2}}\right)\right]>0, \qquad (2.5)$$

then $F(z)=\left(f_1(z)f_2(z)\right)^{e^{-i\lambda}}\in S^*_{LH}(k)$

Proof. Employing an analogous approach to the derivation of Equation (2.3) from Theorem 2.1, we obtain

$$|\hat{\mu}(z)| = |e^{2i\lambda}| \left| \frac{\mu_1 \frac{(f_1)_z}{f_1} + \mu_2 \frac{(f_2)_z}{f_2}}{\frac{(f_1)_z}{f_1} + \frac{(f_2)_z}{f_2}} \right| \quad (2.6)$$

$$= |e^{2i\lambda}| \left| \frac{\mu_1 \left(1 + \frac{zh_1'}{h_1}\right) + \mu_2 \left(1 + \frac{zh_2'}{h_2}\right)}{1 + \frac{zh_1'}{h_1} + 1 + \frac{zh_2'}{h_2}} \right|$$

We aim to demonstrate that $|\hat{\mu}(z)| \le k$. By assumption, we possess the following

$$k^2 \left| 1 + \frac{zh_1'}{h_1} + 1 + \frac{zh_2'}{h_2} \right|^2 - \left| \mu_1 \left(1 + \frac{zh_1'}{h_1}\right) + \mu_2 \left(1 + \frac{zh_2'}{h_2}\right) \right|^2 \quad (2.7)$$

$$= \left(k^2 - |\mu_1|^2\right) \left| 1 + \frac{zh_1'}{h_1} \right|^2 + \left(k^2 - |\mu_2|^2\right) \left| 1 + \frac{zh_2'}{h_2} \right|^2$$

$$+2\mathrm{Re}\left[\left(k^2 - \mu_1 \overline{\mu_2}\right) \left(1 + \frac{zh_1'}{h_1}\right) \left(\overline{1 + \frac{zh_2'}{h_2}}\right) \right] > 0$$

Therefore, $|\hat{\mu}(z)| \le k$, which implies that F is a K-quasi-conformal log-harmonic mapping.

Moreover, by following a similar proof to that in Theorem 2.1, we find that

$$\mathrm{Re}\left(\frac{zF_z - \overline{z}F_{\overline{z}}}{F} \right) > 0.$$

Hence, F is starlike, and we deduce that $F \in S_{LH}^*(k)$. The proof is completed.

Previous studies have utilized analogous techniques to investigate the properties of har- monic mapping products, as referenced in [9, 14].

Building on this foundation, we now present a theorem that further explores the properties of log-harmonic mappings under specific conditions:

Theorem 2.4. Let $f_j(z) = zh_j(z)\overline{g_j(z)} \in \mathrm{SP}_{LH}(\lambda, k, \phi_j)$ with respect to $\mu_j(z) \in B_0\, (j = 1, 2)$ and $zh_j(z)g_j(z) = \phi_j(z)$, where λ be a real number such that $|\lambda| < \frac{\pi}{2}$. If

$$\phi_j(z) = zh_j(z)g_j(z) = z\exp\left(\int_0^z \frac{2\mu_j(s)}{s(1 - \mu_j(s))} ds \right), \quad (2.8)$$

then $F(z) = \left(f_1(z) f_2(z) \right)^{e^{-i\lambda}} \in S_{LH}^*(k)$.

Proof. By Lemma 1.4, substituting (2.8) into (1.5), we obtain

$$1 + \frac{zh_j'(z)}{h_j(z)} = \frac{1}{1 - \mu_j(z)} \quad (j = 1, 2). \quad (2.9)$$

Using a similar argument to that in Theorem 2.3,substituting (2.9) into (2.7), we find that

- $$k^2\left|\frac{1}{1-\mu_1}+\frac{1}{1-\mu_2}\right|^2-\left|\mu_1\left(\frac{1}{1-\mu_1}\right)+\mu_2\left(\frac{1}{1-\mu_2}\right)\right|^2 \qquad (2.10)$$

- $$=\left(k^2-|\mu_1|^2\right)\left|\frac{1}{1-\mu_1}\right|^2+\left(k^2-|\mu_2|^2\right)\left|\frac{1}{1-\mu_2}\right|^2+2\,\mathrm{Re}\left[\frac{k^2-\mu_1\overline{\mu_2}}{(1-\mu_1)\left(1-\overline{\mu_2}\right)}\right].$$

Let $\mu_j=\rho_j e^{i\theta_j},(0\le\rho_j<1,\theta_j\in\mathbb{R};\ j=1,2)$. Thus, it suffices to show that

- $$\mathrm{Re}\left[\frac{k^2-\mu_1\overline{\mu_2}}{(1-\mu_1)\left(1-\overline{\mu_2}\right)}\right]\ge 0$$

Implies $|\hat{\mu}(z)|\le k$ .Since

- $$\mathrm{Re}\left[\frac{k^2-\mu_1\overline{\mu_2}}{(1-\mu_1)\left(1-\overline{\mu_2}\right)}\right]=\frac{1}{|1-\mu_1|^2\left|1-\overline{\mu_2}\right|^2}\mathrm{Re}\left[\left(k^2-\mu_1\overline{\mu_2}\right)\left(1-\overline{\mu_1}\right)(1-\mu_2)\right]$$

- $$=\frac{1}{|1-\mu_1|^2\left|1-\overline{\mu_2}\right|^2}\left(\left(k^2-\rho_1^2\rho_2^2\right)-\rho_1\left(k^2-\rho_2^2\right)\cos\theta_1\right.$$

- $$\left.-\rho_2\left(k^2-\rho_1^2\right)\cos\theta_2-\rho_1\rho_2\left(1-k^2\right)\cos\left(\theta_2-\theta_1\right)\right)$$

- $$\ge\frac{1}{|1-\mu_1|^2\left|1-\overline{\mu_2}\right|^2}\left(\left(k^2-\rho_1^2\rho_2^2\right)-\rho_1\left(k^2-\rho_2^2\right)-\rho_2\left(k^2-\rho_1^2\right)\right.$$

- $$\left.-\rho_1\rho_2\left(1-k^2\right)\right)$$

- $$=\frac{1}{|1-\mu_1|^2\left|1-\overline{\mu_2}\right|^2}\left(k^2-\rho_1\rho_2\right)(1-\rho_1)(1-\rho_2)\ge 0$$

Therefore, $|\hat{\mu}(z)|\le k$, which implies that F is a K-quasi-conformal log-harmonic mapping.

Moreover, by following a similar proof to that in Theorem 2.1, we find that

- $$\mathrm{Re}\left(\frac{zF_z-\overline{z}F_{\overline{z}}}{F}\right)>0.$$

Hence, F is starlike, and we deduce that $F\in S^*_{LH}(k)$. The proof is completed.

The following corollary is an immediate consequence of Theorem 2.4.

Corollary 2.5. Let $f_j(z)=zh_j(z)\overline{g_j(z)}\in SP_{LH}(\lambda,k,\phi_j)$ with respect to the $\mu_j(z)\in B_0\,(j=1,2)$ and $zh_j(z)g_j(z)=\phi_j(z)$, where λ be a real number such that $|\lambda|<\frac{\pi}{2}$. If

- $$\phi_j(z)=zh_j(z)g_j(z)=z\exp\left(\int_0^z\frac{t(1+\mu_j(s))+\mu_j^n(s)-1}{s(1-\mu_j^n(s))}\right)ds \quad (n\ge 1,0<t\le 1)$$

then $F(z)=\left(f_1(z)f_2(z)\right)^{e^{-i\lambda}}\in S^*_{LH}(k)$.

Proof. Since $zh_j(z)g_j(z)=\phi_j(z)$, by definition of $\mu_j(z)$, we obtain

- $$1+\frac{zh_j'(z)}{h_j(z)}=\frac{t}{1-\mu_j^n(z)} \quad (j=1,2).$$

Using a similar argument as in Theorem 2.4, we only need to prove that

- $$\mathrm{Re}\left[\frac{k^2-t^2\mu_1\overline{\mu_2}}{\left(1-\mu_1^n\right)\left(1-\overline{\mu_2^n}\right)}\right]>0.$$

Since

- $$\mathrm{RE}\left[\frac{k^2-t^2\mu_1\overline{\mu_2}}{\left(1-\mu_1^n\right)\left(1-\overline{\mu_2^n}\right)}\right]=\frac{1}{\left|1-\mu_1^n\right|^2\left|1-\overline{\mu_2^n}\right|^2}\mathrm{RE}\left[\left(k^2-t^2\mu_1\overline{\mu_2}\right)\left(1-\overline{\mu_1^n}\right)\left(1-\mu_2^n\right)\right]$$
- $$=\frac{1}{\left|1-\mu_1^n\right|^2\left|1-\overline{\mu_2^n}\right|^2}\Big(\Big(k^2-t^2\rho_1^{n+1}\rho_2^{n+1}\cos\left(\left(n-1\right)\left(\theta_1-\theta_2\right)\right)\Big)$$
- $$+\rho_1^n\left(t^2\rho_1\rho_2\cos\left(\left(1-n\right)\theta_1-\theta_2\right)-k^2\cos\left(n\theta_1\right)\right)$$
- $$+\rho_2^n\left(t^2\rho_1\rho_2\cos\left(\theta_1-\left(1-n\right)\theta_2\right)-k^2\cos\left(n\theta_2\right)\right)$$
- $$-\rho_1\rho_2\left(t^2\cos\left(\theta_1-\theta_2\right)+k^2\rho_1^{n-1}\rho_2^{n-2}\cos\left(n\left(\theta_2-\theta_1\right)\right)\right)\Big)$$

$$\geq\frac{1}{\left|1-\mu_1^n\right|^2\left|1-\overline{\mu_2^n}\right|^2}\Big(\left(k^2-t^2\rho_1^{n+1}\rho_2^{n+1}\right)-\rho_1^n\left(k^2+t^2\rho_1\rho_2\right)$$

- $$-\rho_2^n\left(k^2+t^2\rho_1\rho_2\right)-\rho_1\rho_2\left(t^2+k^2\rho_1^{n-1}\rho_2^{n-1}\right)\Big)$$
- $$=\frac{1}{\left|1-\mu_1^n\right|^2\left|1-\overline{\mu_2^n}\right|^2}\left(k^2-t^2\rho_1\rho_2\right)\left(1-\rho_1^n\right)\left(1-\rho_2^n\right)\geq 0$$

Therefore, $\left|\hat{\mu}(z)\right|\leq k$, which implies that F is a K-quasi-conformal log-harmonic mapping.

Moreover, by following a similar proof to that in Theorem 2.1, we find that

- $$\mathrm{Re}\left(\frac{zF_z-\overline{z}F_{\overline{z}}}{F}\right)>0.$$

Hence, F is starlike, and we deduce that $F\in S_{LH}^*(k)$. The proof is completed.

Theorem 2.6. Let $f_j(z)=zh_j(z)\overline{g_j(z)}\in \mathrm{SP}_{LH}(\lambda,k,\phi_j)$ with respect to $\mu_j(z)\in \mathrm{B}_0\,(j=1,2)$ and $zh_j(z)g_j(z)=\phi_j(z)$, where λ be a real number such that $|\lambda|<\frac{\pi}{2}$. If

- $$\phi_j(z)=zh_j(z)g_j(z)=z,$$

then $F(z)=\left(f_1(z)f_2(z)\right)^{e^{-i\lambda}}\in S^*_{LH}(k)$.

Proof: Since $zh_j(z)g_j(z)=\phi_j(z)$, by the definition of $\mu_j(z)$, we obtain

- $1+\dfrac{zh_j'(z)}{h_j(z)}=\dfrac{1}{1+\mu_j(z)}\quad (j=1,2).$

By employing a similar argument as in Theorem 2.4, we have $|\hat{\mu}(z)|\le k$ if

- $k^2\left|\dfrac{1}{1+\mu_1}+\dfrac{1}{1+\mu_2}\right|^2-\left|\mu_1\left(\dfrac{1}{1+\mu_1}\right)+\mu_2\left(\dfrac{1}{1+\mu_2}\right)\right|^2$
- $=\left(k^2-|\mu_1|^2\right)\left|\dfrac{1}{1+\mu_1}\right|^2+\left(k^2-|\mu_2|^2\right)\left|\dfrac{1}{1+\mu_2}\right|^2+2\operatorname{Re}\left[\dfrac{k^2-\mu_1\overline{\mu_2}}{(1+\mu_1)(1+\overline{\mu_2})}\right].$

Let $\mu_j=\rho_je^{i\theta_j},(0\le\rho_j<1,\theta_j\in\mathbb{R};\ j=1,2)$. Thus, it suffices to show that

- $\operatorname{Re}\left[\dfrac{k^2-\mu_1\overline{\mu_2}}{(1+\mu_1)(1+\overline{\mu_2})}\right]\ge 0$

Implies $|\hat{\mu}(z)|\le k$ .Since

- $\operatorname{Re}\left[\dfrac{k^2-\mu_1\overline{\mu_2}}{(1+\mu_1)(1+\overline{\mu_2})}\right]=\dfrac{1}{|1+\mu_1|^2|1+\overline{\mu_2}|^2}\operatorname{Re}\left[\left(k^2-\mu_1\overline{\mu_2}\right)\left(1+\overline{\mu_1}\right)(1+\mu_2)\right]$
- $=\dfrac{1}{|1+\mu_1|^2|1+\overline{\mu_2}|^2}\Big(\left(k^2-\rho_1^2\rho_2^2\right)-\rho_1\left(\rho_2^2-k^2\right)\cos\theta_1$
- $-\rho_2\left(\rho_1^2-k^2\right)\cos\theta_2-\rho_1\rho_2\left(1-k^2\right)\cos\left(\theta_2-\theta_1\right)\Big)$
- $\ge\dfrac{1}{|1+\mu_1|^2|1+\overline{\mu_2}|^2}\Big(\left(k^2-\rho_1^2\rho_2^2\right)-\rho_1\left(k^2-\rho_2^2\right)-\rho_2\left(k^2-\rho_1^2\right)$
- $-\rho_1\rho_2\left(1-k^2\right)\Big)$
- $=\dfrac{1}{|1+\mu_1|^2|1+\overline{\mu_2}|^2}\left(k^2-\rho_1\rho_2\right)(1-\rho_1)(1-\rho_2)\ge 0$

Therefore, $|\hat{\mu}(z)|\le k$, which implies that F is a K-quasi-conformal log-harmonic mapping.

Moreover, by following a similar proof to that in Theorem 2.1, we find that

- $\operatorname{Re}\left(\dfrac{zF_z-\overline{z}F_{\overline{z}}}{F}\right)>0.$

Hence, F is starlike, and we deduce that $F\in S^*_{LH}(k)$. The proof is completed.

The following corollary is an immediate consequence of Theorem 2.4.

Corollary 2.7. Let $f_j(z)=zh_j(z)\overline{g_j(z)}\in SP_{LH}(\lambda,k,\phi_j)$ with respect to the $\mu_j(z)\in B_0\,(j=1,2)$ and $zh_j(z)g_j(z)=\phi_j(z)$, where λ be a real number such that $|\lambda|<\frac{\pi}{2}$. If

- $\phi_j(z)=zh_j(z)g_j(z)=z\exp\left(\int_0^z \frac{t(1+\mu_j(s))-\mu_j^n(s)-1}{s(1+\mu_j^n(s))}ds\right) \quad (n\geq 1, 0<t\leq 1)$

then $F(z)=\left(f_1(z)f_2(z)\right)^{e^{-i\lambda}}\in S_{LH}^*(k)$.

Proof. Since $zh_j(z)g_j(z)=\phi_j(z)$, by definition of $\mu_j(z)$, we obtain

- $1+\frac{zh_j'(z)}{h_j(z)}=\frac{t}{1+\mu_j^n(z)} \quad (j=1,2).$

Using a similar argument as in Theorem 2.4, we only need to prove that

- $\operatorname{Re}\left[\frac{k^2-t^2\mu_1\overline{\mu_2}}{\left(1+\mu_1^n\right)\left(1+\overline{\mu_2^n}\right)}\right]>0.$

Since

- $\operatorname{RE}\left[\frac{k^2-t^2\mu_1\overline{\mu_2}}{\left(1+\mu_1^n\right)\left(1+\overline{\mu_2^n}\right)}\right]=\frac{1}{\left|1+\mu_1^n\right|^2\left|1+\overline{\mu_2^n}\right|^2}\operatorname{RE}\left[\left(k^2-t^2\mu_1\overline{\mu_2}\right)\left(1+\overline{\mu_1^n}\right)\left(1+\mu_2^n\right)\right]$
- $=\frac{1}{\left|1+\mu_1^n\right|^2\left|1+\overline{\mu_2^n}\right|^2}\Big(\Big(k^2-t^2\rho_1^{n+1}\rho_2^{n+1}\cos\left((n-1)(\theta_1-\theta_2)\right)\Big)$
- $-\rho_1^n\left(t^2\rho_1\rho_2\cos\left((1-n)\theta_1-\theta_2\right)-k^2\cos\left(n\theta_1\right)\right)$
- $-\rho_2^n\left(t^2\rho_1\rho_2\cos\left(\theta_1-(1-n)\theta_2\right)-k^2\cos\left(n\theta_2\right)\right)$
- $-\rho_1\rho_2\left(t^2\cos\left(\theta_1-\theta_2\right)+k^2\rho_1^{n-1}\rho_2^{n-2}\cos\left(n\left(\theta_2-\theta_1\right)\right)\right)\Big)$

$\geq\frac{1}{\left|1+\mu_1^n\right|^2\left|1+\overline{\mu_2^n}\right|^2}\Big(\left(k^2-t^2\rho_1^{n+1}\rho_2^{n+1}\right)-\rho_1^n\left(k^2+t^2\rho_1\rho_2\right)$

- $-\rho_2^n\left(k^2+t^2\rho_1\rho_2\right)-\rho_1\rho_2\left(t^2+k^2\rho_1^{n-1}\rho_2^{n-1}\right)\Big)$
- $=\frac{1}{\left|1+\mu_1^n\right|^2\left|1+\overline{\mu_2^n}\right|^2}\left(k^2-t^2\rho_1\rho_2\right)\left(1-\rho_1^n\right)\left(1-\rho_2^n\right)\geq 0$

Therefore, $|\hat{\mu}(z)|\leq k$, which implies that F is a K-quasi-conformal log-harmonic mapping. Following the proof structure of Theorem 2.1, we conclude that

- $\operatorname{Re}\left(\frac{zF_z-\overline{z}F_{\overline{z}}}{F}\right)>0.$

Hence, F is starlike, and we deduce that $F\in S_{LH}^*(k)$. The proof is completed.

Theorem 2.8. Let $f_j(z)=zh_j(z)\overline{g_j(z)}\in SP_{LH}(\lambda,k,\phi_j)$ with respect to $\mu_j(z)\in B_0(j=1,2)$ and $zh_j(z)g_j(z)=\phi_j(z)$, where λ be a real number such that $|\lambda|<\frac{\pi}{2}$. If

- $$\phi_1(z)=zh_1(z)g_1(z)=z\exp\left(\int_0^z\frac{2\mu_1(s)}{s(1-\mu_1(s))}ds\right),$$

$$\phi_j(z)=zh_j(z)g_j(z)=z,$$

then $F(z)=\left(f_1(z)f_2(z)\right)^{e^{-i\lambda}}\in S^*_{LH}(k)$.

Proof. Since $zh_j(z)g_j(z)=\phi_j(z)$, by definition of $\mu_j(z)$, we obtain

- $$1+\frac{zh_1'(z)}{h_1(z)}=\frac{1}{1-\mu_1(z)}.$$

$$1+\frac{zh_2'(z)}{h_2(z)}=\frac{1}{1+\mu_2(z)}.$$

Applying the same reasoning as in Theorem 2.6, it is sufficient to establish that

- $$\mathrm{Re}\left[\frac{k^2-\mu_1\overline{\mu_2}}{(1-\mu_1)(1+\overline{\mu_2})}\right]\geq 0$$

To this end, we consider

- $$\mathrm{Re}\left[\frac{k^2-\mu_1\overline{\mu_2}}{(1-\mu_1)(1+\overline{\mu_2})}\right]=\frac{1}{|1-\mu_1|^2|1+\overline{\mu_2}|^2}\mathrm{Re}\left[\left(k^2-\mu_1\overline{\mu_2}\right)\left(1-\overline{\mu_1}\right)\left(1+\mu_2\right)\right]$$
- $$=\frac{1}{|1-\mu_1|^2|1+\overline{\mu_2}|^2}\left(\left(k^2+\rho_1^2\rho_2^2\right)-\rho_1\left(k^2+\rho_2^2\right)\cos\theta_1\right.$$
- $$\left.+\rho_2\left(k^2-\rho_1^2\right)\cos\theta_2-\rho_1\rho_2\left(1-k^2\right)\cos\left(\theta_2-\theta_1\right)\right)$$
- $$\geq\frac{1}{|1-\mu_1|^2|1+\overline{\mu_2}|^2}\left(\left(k^2-\rho_1^2\rho_2^2\right)-\rho_1\left(k^2+\rho_2^2\right)-\rho_2\left(k^2+\rho_1^2\right)\right.$$
- $$\left.-\rho_1\rho_2\left(1+k^2\right)\right)$$
- $$=\frac{1}{|1-\mu_1|^2|1+\overline{\mu_2}|^2}\left(k^2-\rho_1\rho_2\right)\left(1-\rho_1\right)\left(1+\rho_2\right)\geq 0$$

Therefore, $|\hat{\mu}(z)|\leq k$, which implies that F is a K-quasi-conformal log-harmonic mapping. Following the proof structure of Theorem 2.1, we conclude that

- $$\mathrm{Re}\left(\frac{zF_z-\overline{z}F_{\overline{z}}}{F}\right)>0.$$

Hence, F is starlike, and we deduce that $F \in S^{*}_{LH}(k)$. The proof is completed.

Corollary 2.9. Let $f_j(z) = zh_j(z)\overline{g_j(z)} \in SP_{LH}(\lambda, k, \phi_j)$ with respect to $\mu_j(z) \in B_0 (j = 1, 2)$ and $zh_j(z)g_j(z) = \phi_j(z)$, where λ be a real number such that $|\lambda| < \frac{\pi}{2}$. If

- $\phi_j(z) = zh_j(z)g_j(z) = z\exp\left(\int_0^z \frac{t(1+\mu_j(s)) - \mu_j^n(s) - 1}{s(1-\mu_j^n(s))}ds\right) \quad (n \ge 1, 0 < t \le 1)$

then $F(z) = \left(f_1(z)f_2(z)\right)^{e^{-i\lambda}} \in S^{*}_{LH}(k)$.

Proof. Since $zh_j(z)g_j(z) = \phi_j(z)$, by definition of $\mu_j(z)$, we obtain

- $1 + \frac{zh_j'(z)}{h_j(z)} = \frac{t}{1-\mu_j^n(z)} \quad (j = 1, 2).$

As reasoned in Corollary 2.7, $F(z)$ satisfies the conditions for being a starlike log-harmonic mapping in $S^{*}_{LH}(k)$. The detailed steps are analogous to those in the referenced corollary, and thus are omitted for brevity.

Theorem 2.10. Let $f_j(z) = zh_j(z)\overline{g_j(z)} \in SP_{LH}(\lambda, k, \alpha_j)$ for $j = 1, 2$ with respect to the same $\mu(z) \in B_0$, where λ be a real number such that $\frac{\pi}{3} < |\lambda| < \frac{\pi}{2}$. Then

- $F(z) = \left(f_1(z)f_2(z)\right)^{e^{-i\lambda}} \in S^{*}_{LH}(k, (\alpha_1 + \alpha_2)\cos\lambda)$

Proof. Drawing parallels with the approach in Theorem 2.1, we establish that $|\hat{\mu}(z)| \le k$. To further substantiate our claim, we need to demonstrate that

- $\mathrm{Re}\left(\frac{zF_z - \overline{z}F_{\overline{z}}}{F}\right) = \mathrm{Re}\left(e^{-i\lambda}\frac{z(f_1)_z - \overline{z}(f_1)_{\overline{z}}}{f_1}\right) + \mathrm{Re}\left(e^{-i\lambda}\frac{z(f_2)_z - \overline{z}(f_2)_{\overline{z}}}{f_2}\right)$
- $\ge \alpha_1\cos\lambda + \alpha_2\cos\lambda = (\alpha_1 + \alpha_2)\cos\lambda$

Given the constraint

- $0 \le (\alpha_1 + \alpha_2)\cos\lambda < 1 \Rightarrow \frac{\pi}{3} < |\lambda| < \frac{\pi}{2}.$

We infer that $\frac{\pi}{3} < |\lambda| < \frac{\pi}{2}$, which aligns with the given condition on λ . This inequality ensures that the real part of the expression is positive and less than unity, confirming that $F(z)$ is indeed starlike with the desired properties.

Corollary 2.11. Let $f_j(z) = zh_j(z)\overline{g_j(z)} \in SP_{LH}(\lambda, k, \alpha_j)$ with respect to the same $\mu(z) \in B_0$, where λ be a real number such that $\arccos\frac{1}{n} < |\lambda| < \frac{\pi}{2}$. Then

- $F(z) = \left(f_1(z)f_2(z)\cdots f_n(z)\right)^{e^{-i\lambda}} \in S^{*}_{LH}(k, (\alpha_1 + \alpha_2 + \cdots + \alpha_n)\cos\lambda)$

Proof. As established in Theorem 2.1, we have $|\hat{\mu}(z)| \le k$. Building upon this foundation, we aim to demonstrate that

$$\operatorname{Re}\left(\frac{zF_z-\bar{z}F_{\bar{z}}}{F}\right)=\operatorname{Re}\left(e^{-i\lambda}\frac{z(f_1)_z-\bar{z}(f_1)_{\bar{z}}}{f_1}\right)+\operatorname{Re}\left(e^{-i\lambda}\frac{z(f_2)_z-\bar{z}(f_2)_{\bar{z}}}{f_2}\right)$$

$$+\cdots+\operatorname{Re}\left(e^{-i\lambda}\frac{z(f_n)_z-\bar{z}(f_n)_{\bar{z}}}{f_n}\right)$$

$$\geq \alpha_1\cos\lambda+\alpha_2\cos\lambda+\cdots+\alpha_n\cos\lambda$$

$$=(\alpha_1+\alpha_2+\cdots+\alpha_n)\cos\lambda$$

Given the constraint $0\leq(\alpha_1+\alpha_2+\cdots+\alpha_n)\cos\lambda<1$, it directly follows that $\arccos\frac{1}{n}<|\lambda|<\frac{\pi}{2}$,which aligns with the conditions provided for λ . This completes the proof that $F(z)$ satisfies the required properties to be an element of $S^*_{LH}(k,(\alpha_1+\alpha_2+\cdots+\alpha_n)\cos\lambda)$.

Further advancements in convex-exponent properties and Schwarzian derivatives of log-harmonic mappings, as discussed in [5, 8], could extend the results presented in this paper.

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