Current Science
CS 3 (1), 284– 291 (2023)
Received 20 Nov 2022 | **Revised** 15 Dec 2022 | **Accepted** 25 Jan 2023 | **Online Available** 06 Feb 2023

Current Science
Current Science

OPEN ACCESS

ISSN(O)2795-8639

ORIGINAL ARTICLE

3D Visualization of the Exact Womersley Velocity Profile

Arati Nanda Pati

Department of Computer Science, Rice University, Houston, TX 77005

Corresponding Author: Arati Nanda Pati

Abstract: The three-dimensional (3D) visualization of Womersley flow can provide important physiological and physical insights into the Biofluid dynamics of the blood flow through an artery, which is of great significance in the understanding and treatment of vascular diseases, like heart attack and stroke. In this paper, our goal is to visualize the unsteady velocity profile of the exact Womersley solution obtained in a circular channel due to varying pressure gradients at the inlet and outlet that change with time. In contrast to a steady flow of parabolic nature, the analytical solution of unsteady flow obtained using Bessel functions has complex dynamics. Using a fictitious method, we developed a circular cylinder from a rectangular parallelepiped to better conceptualize these flow patterns with 3D graphics and animation. We trust this work would be a primer to better understand blood flow due to time-dependent aortic and ventricular pressure differences which have direct implications in clinical practice.

Keyterms: Womersley number, unsteady flow, Bessel functions, fictitious method

Introduction

The use of mathematical models to describe natural events provides an additional way to understand and comprehend their behavior and physiology. Arterial blood pressure has been studied initially using Poiseuille's law. In 1955, Womersley while working with McDonald published his famous equations of oscillating hemodynamics of blood flow in the arterial system [1-5]. McDonald was a physiologist and Womersley was trying to develop a set of equations (constitutive equations) to predict McDonald's findings. The work was done in collaboration with McDonald and Taylor by evaluating the alternating pressure gradients observed in the arterial system [6,7]. In subsequent years his work was further validated as an adequate representation of the flow mechanism in arteries [8-9].

The formula proposed by Womersley pursued to simplify and create a realistic model of the cardiovascular system. To perform this, it was necessary to determine the differential equations that described the model, solve the equations, and determine any unknown constants used. When pressure gradient is constant, a steady flow with parabolic velocity is obtained in a circular cylinder. However, a complex velocity pattern is observed when the pressure gradient is not a constant and varies with time. A parameter which characterized the nature of unsteady flow, called the Womersley number represented by α, plays an important role. The use of this number allows for the calculation of flow in a rigid tube that increases and decreases in conjunction with an alternating pressure gradient [10].

The purpose of this paper is to reconstruct the exact Womersley 3D computational model and visualize it as a function of time to have a space-time 4D visualization for a better understanding of the complex mathematical formula.

Method

For a non-constant pressure gradient expressed in the form

$$\frac{p \quad - p_2}{H} = Ae^{i\omega t}$$

Womersley derived the following exact solution, equation (9) to the unsteady flow shown in the appendix.

$$w(r,t) = \frac{A}{\rho}\frac{1}{i\omega}\left[1 - \frac{J_0\left(\alpha y i^{\frac{3}{2}}\right)}{J_0\left(\alpha i^{\frac{3}{2}}\right)}\right] e^{i\omega t}$$

Here, J_0 is the Bessel function of order zero of 1st Kind. Under special consideration, it reduces to equation (10).

$$w(r,t) = \frac{M}{\rho}\frac{1}{\omega}\left[\sin(\omega t + \phi) - \frac{M_0(y)}{M_0}\sin(\omega t + \phi - \delta_0\right]$$

Our goal is to visualize the above equation (10) and better understand the meaning of this mathematical equation that changes with time.

To display the result, we consider a 3D circular cylinder with a diameter of 1. To create it computationally using the fictitious method, we assume that the circular cylinder has been embedded in a rectangular region whose sides are a little bigger than 1 as shown in Figure 1.

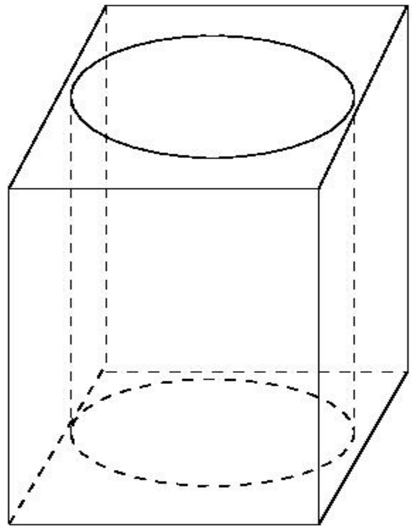


Figure 1. Cylindrical flow region within a square parallelepied

Assume that one end of the diameter of the circle at the base is at 0 and the other end is at 1. To create a little bigger side, we extend a small amount at both ends which can completely include the diameter 1 of the circle. We also assume the height of the computational domain to be H=2. Additionally, we consider the 3D rectangular region is filled with fluid with *kinematic viscosity* $\nu = 0.1$. The fluid flows due to pressure difference at both ends that depends on time t where outside the circular region fluid flow is zero. In order to create unsteady flow, we assume sinusoidal pressure p=AHcos ωt where A=-40 is the amplitude, H is the height of the channel, ω is the angular frequency, and t is the time at one end and P=0 at the other end. From Equation (7) we calculate the angular frequency for a given *Womersley number*, α. Corresponding to our pressure gradient in the form of cosine, a modified version of the exact solution, Equation (9) can be derived in Pati [11]. We will be able to visualize the unsteady exact solution described in the results and discussion section. In [11], Pati et al. have compared the exact solution to oscillatory flow with a numerical solution by applying Distributed Lagrange multiplier method.

We use Fortran 95 to write a computer code to determine the flow characteristics due to pressure gradient. Later we use MATLAB to visualize the flow in a three-dimensional domain.

Results and Discussion:

When the pressure gradient is constant, flow in a cylinder remains steady with a parabolic velocity profile with the maximum velocity at the center and gradually decreases to zero at the boundary. However, when the pressure gradient is not constant, a complex flow pattern is observed. The fluid flow slows down, stops and reverses direction, accelerates in that direction, and then slows down again.

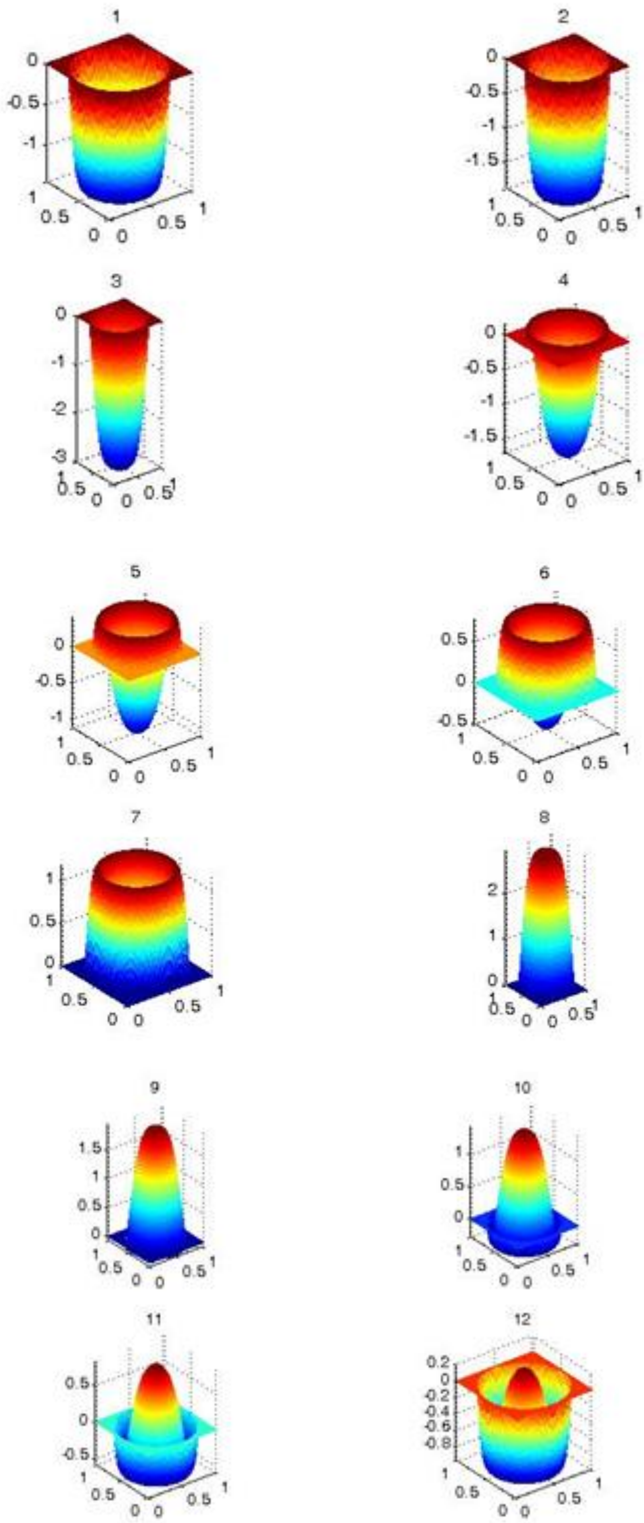


Figure 2. Womersley exact velocity profile at mid plane of the 3D circular cylinder for α=6. From left to right as we go down each row the profiles are given at time t=0.015, 0.030, 0.120, 0.180, 0.195, 0.210, 0.225, 0.345, 0.390, 0.405, 0.420, 0.435.

In Figure 2, we have visualized the exact velocity profile at the midplane of the 3D circular cylinder for α=6 for one time period at different times. As we move from left to right and top to bottom, we see the velocity profiles are not parabolic and gradually flow reverses directions which are dominant near the boundary of the circle than the center. Since our computational region was rectangular, we observe that the flow outside the circle is zero. A close-up view of velocity profiles is visualized in Figures 3 and 4.

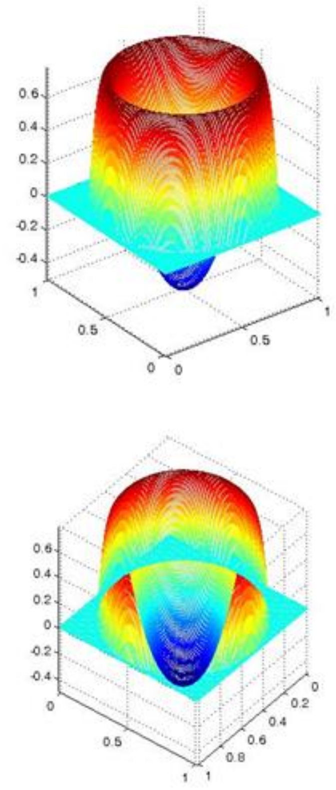


Figure 3. Top: Womersley exact velocity profile at mid plane of the 3D circular cylinder at time t= 0.210 for α=6, (view from top). Bottom: Same picture with rotation (view from different angle).

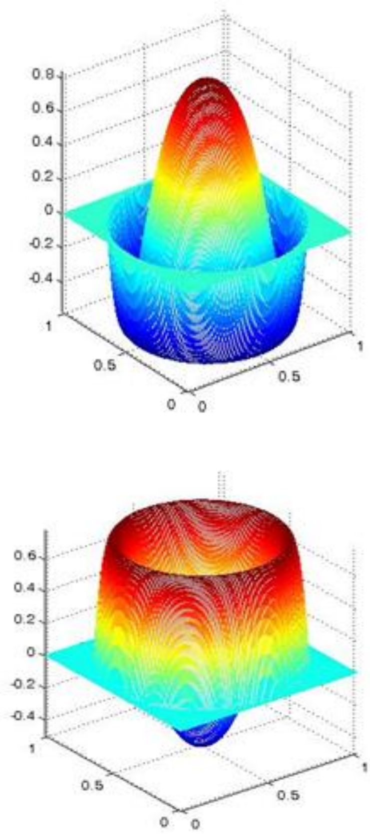


Figure 4. Top: Womersley exact velocity profile at mid plane of the 3D circular cylinder at time t = 0.420 for α=6. Bottom: Same picture with rotation (view from different angle).

In order to have a view of individual shapes of the velocity profile along the horizontal line we have shown velocity profiles at the middle of the channel along the x-axis for one time period in Figure 5, where we observe pattern change from forward to backward and center to the boundary.

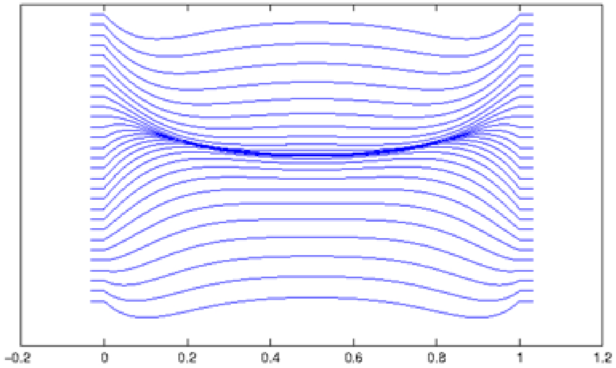


Figure 5. Exact oscillating flow profiles at mid diagonal along x-axis (x,0.5,1) of the 3D circular cylinder during one time period from t=0.015 to t=0.435 in every 15 time steps for α=6.

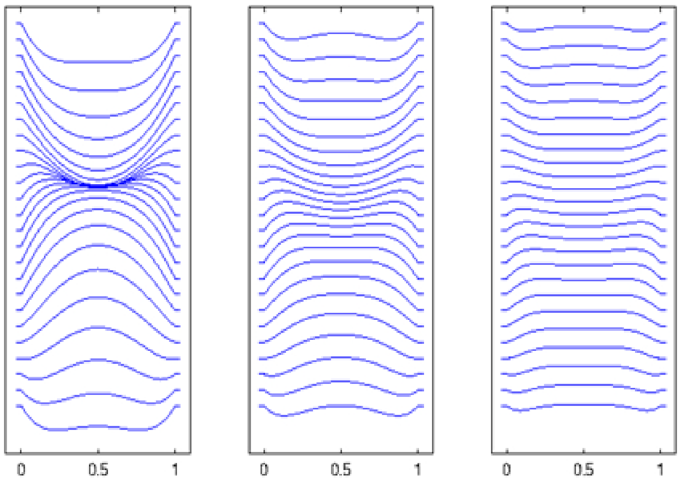
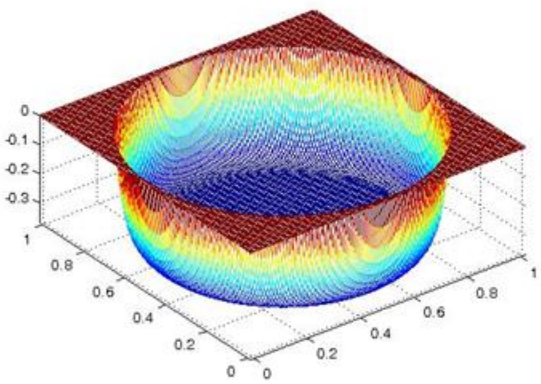


Figure 6. Exact oscillating flow profiles at mid diagonal along x-axis (x,0.5,1) of the 3D circular cylinder during one time period for α=3.34 (left), α=4.72 (middle), α=6.67 (right).

Figure 6 compares shapes of velocity profiles for different values of Womersley number α. It suggests that as we increase the value, the profiles become flatter at the center as shown in Figure 7.



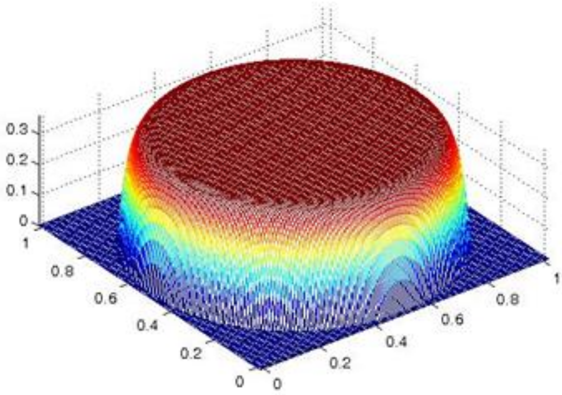


Figure 7. Womersley exact velocity profile at mid plane of the 3D circular cylinder for α=16.4. Top: at time t=1.6 × 10^{-2}, Bottom: at time t=4.59 × 10^{-2}.

In summary, we have shown that the Womersley number plays an important role in unsteady flow in a circular cylinder. We have visualized the flow patterns for exact solutions with time. To have a better understanding, we have attached an animation for α=6 for the exact solution of unsteady flow in the supplementary folder.

https://drive.google.com/drive/folders/1Q19IqKiXOkUXZfEAHpvs249xYXx5bUyB

References

1. Womersley J. R. Flow in the larger arteries and its relation to the oscillating pressure. J Physiol. 124(Suppl);1954: 31P-32P.
2. Womersley J. R. Mathematical theory oscillating flow in an elastic tube. J Physiol. 127(Suppl);1955: 37P-38P.
3. Womersley J. R. Oscillatory flow in arteries: effect of radial variation in viscosity on rate of flow. J Physiol. 127(Suppl);1955: 38P-39P.
4. Womersley J. R. Method for calculation of velocity, rate of flow and viscous drag in arteries when the pressure gradient is known. J Physiol. 127;1955: 553-563.
5. McDonald D. A. The relation of pulsatile pressure to flow in arteries. J Physiol 127; 1955: 533-552.
6. McDonald D. A. Blood flow in arteries. Arnold, London, 1998.
7. Womersley J. R. Oscillatory flow in arteries: the constrained elastic tube as a model of arterial flow and pulse transmission. Phys. Med. Biol. 1957;2:178-187.
8. Womersley J. R. Oscillatory flow in arteries II: The reflections of pulse wave at junctions and rigid inserts in the arterial system. Phys. Med. Biol. 1958;2:313-323.
9. Womersley J. R. Oscillatory flow in arteries III: Flow and Pulse-velocity formulae for a liquid whose viscosity varies with frequency. Phys. Med. Biol. 1958;2:374-382.
10. Caro C. G, Pedley T. J, Schroter R. C, Seed W. A. The mechanics of the circulation. Oxford University Pres, New York, 1978; 57-60.
11. Pati A. N, Ladipo K, Paniagua D, Glowinski R. Three Dimensional Fluid-Structure Interaction under Pulsatile Flow by using Distributed Lagrange Multiplied Method. Mathematical and Computer Modelling 53, 2011,21-41.
12. McLachlan N. W. Bessel Functions for Engineers, Oxford University Press, 1941.

Appendix

For a non-constant pressure gradient expressed in the form,

$$\frac{p_1 - p_2}{H} = Ae^{i\omega t} \tag{1}$$

the equation of motion for oscillating flow is given by

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} - \frac{1}{\nu}\frac{\partial w}{\partial t} = -\frac{A}{\mu}\, e^{i\omega t} \tag{2}$$

Using the separation of variables

$$w(r,t) = u(r)e^{i\omega t} \tag{3}$$

equation (2) can be written in the ODE form

$$\frac{d^2 u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{i\omega}{\nu}u = -\frac{A}{\mu} \tag{4}$$

This ODE has solutions in terms of Bessel functions [12] given by

$$u(r) = \frac{A}{\rho}\frac{1}{i\omega} + A_c J_0\left(r\sqrt{\frac{\omega}{\nu}}\, i^{\frac{3}{2}}\right) + B_c K_0\left(r\sqrt{\frac{\omega}{\nu}}\, i^{\frac{1}{2}}\right) \tag{5}$$

where A_c and B_c are the constants to be determined from the boundary conditions with J_0, the Bessel function of order zero of the first kind, and K_0, the Bessel function of order zero of the second kind. Here the first term is the particular solution and the second and third terms are the solution of the homogeneous equation of the ODE (4). From the properties of $K_0\left(r\sqrt{\frac{\omega}{\nu}}\, i^{\frac{1}{2}}\right)$, u(r) becomes infinite on the axis of the tube where r=0. But we need finite velocity along the axis of the tube, therefore $B_c = 0$. Due to the no-slip boundary condition at the radius of the cylinder, R, u(R)=0, we can evaluate A_c from (5). Hence the exact solution to the ODE (4) is

$$u(r) = \frac{A}{\rho}\frac{1}{i\omega}\left[1 - \frac{J_0\left(r\sqrt{\frac{\omega}{\nu}}\, i^{\frac{3}{2}}\right)}{J_0\left(R\sqrt{\frac{\omega}{\nu}}\, i^{\frac{3}{2}}\right)}\right] \tag{6}$$

Now introducing the Womersley number, $\propto$ to be

$$\propto = R\sqrt{\frac{\omega}{\nu}} \tag{7}$$

and

$$y = \frac{r}{R} \tag{8}$$

the exact solution (3) for the oscillating flow (2) will be given by

$$w(r,t) = \frac{A}{\rho}\frac{1}{i\omega}\left[1 - \frac{J_0\left(\alpha y i^{\frac{3}{2}}\right)}{J_0\left(\alpha i^{\frac{3}{2}}\right)}\right] e^{i\omega t} \tag{9}$$

The non-dimensional parameters R,r,ν,ρ, and ω are the radius of the circular cylinder, distance from the axis, kinetic viscosity, fluid density, and angular frequency, respectively.

If the given pressure is real part of $Ae^{i\omega t} = M\cos(\omega t + \phi)$ then equation (9) will reduce to

$$w(r,t) = \frac{M}{\rho}\frac{1}{\omega}\left[\ \sin(\omega t+\phi) - \frac{M_0(y)}{M_0}\sin(\omega t+\phi-\delta_0\right] \quad (10)$$

where

$$\frac{M_0(y)}{M_0} = \frac{M_0(\alpha y)}{M_0(\alpha)} = \frac{\sqrt{(ber^2(\alpha y)+bei^2(\alpha y)}}{\sqrt{(ber^2(\alpha)+bei^2(\alpha)}} \quad (11)$$

and

$$\delta_0 = \tan^{-1}\left(\frac{bei\ \alpha}{ber\ \alpha}\right) - \tan^{-1}\left(\frac{bei\ (\alpha y)}{ber\ (\alpha y)}\right) \quad (12)$$

for real and imaginary parts of the Bessel functions ber and bei [12], respectively. In [11], we have derived the 5 terms for the computation of ber and bei.

Bessel function of order ν can be expressed in terms of

$$J_\nu\left(zi^{\frac{3}{2}}\right) = ber_\nu\ z + i\ bei_\nu\ z \quad (13)$$

as well as in the modulus and phase form as

$$J_\nu\left(zi^{\frac{3}{2}}\right) = M_\nu(z)e^{i\theta_\nu(z)} \quad (14)$$

Where

$$M_\nu(z) = \sqrt{ber_\nu^2\ z + bei_\nu^2\ z} \quad (15)$$

and

$$\theta_\nu(z) = \tan^{-1}\left(\frac{bei_\nu z}{ber_\nu}\right) \quad (16)$$